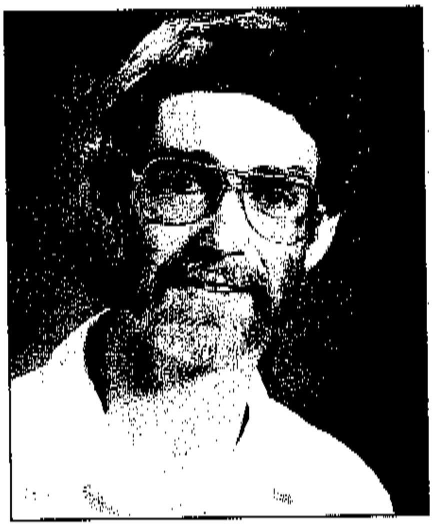
How Does the Wind Affect Road-Running Achievement?

By Michael D. de Villiers



Michael de Villiers *is a lecturer in the Department of Education of the University of Durban-Westville, South Africa. He obtained a B.S. and HDE in 1977 and 1978, respectively, with majors in applied mathematics and physics from the University of Stellenbosch, South Africa. He taught high-school mathematics and science in Karasburg (Namibia) and Kimberley until May 1983, winning a national award as Science Teacher of the Year 1983. From June 1983 to 1990 he worked as researcher at the Research Unit for Mathematics Education at the University of Stellenbosch, during which time he completed a M.Ed. and a D.Ed. and also attended Cornell University (1984–85) on a Rotary Postgraduate Scholarship. He is a keen sportsman, still actively participating in tennis and running. One of his major research interests is the application of school science and mathematics to real-world problems.*

After finishing an out-and-back half marathon in windy conditions, the following typical conversation was overheard:

Runner A: "Heck, that was a good run. If it hadn't been for the strong wind against us going out, I might even have bettered my best time by one or two minutes."

Runner B: "Yes, maybe so, but you must remember that the wind was behind you coming back, making up for the time you lost on the way out: so basically, it cancels out."

Runner A: "I don't know, I still think the wind affects one pretty badly. Maybe one works so hard going out against the wind that one is too tired to benefit from the wind on the way back."

Runner B: "You may be right, but I think any effect of the wind is mostly psychological."

Having myself often been involved in similar situations defending runner A's viewpoint, I know how difficult it can be to convince someone like runner B that the wind plays more than just a psychological role in running. Since road-running participation in South Africa has exploded in recent years, producing household names and public idols like Matthews Temane and Xolile Yawa, world record holders for the half marathon, this problem provides a particularly relevant opportunity for showing South African students the integration of mathematics and physics. One way of examining this problem is to develop a simple algebraic model.

Developing a Mathematical Model

Let us imagine a situation as depicted in Fig. 1, with the average running speed of the runner v km/h on a windless day and a head wind of u km/h as shown. The distance to the turning-around point is d km. Note that we have simplified the situation considerably by assuming a constant running speed (no hills or sandy terrain), a constant wind speed (no gusts or swirling), a straight course, as well as no change in the direction of the wind during the race. Furthermore, we are ignoring the possible influence of the physique of the runner in windy conditions.

We will start with a simple model that is surely too primitive. Let us assume that the resultant speed is the sum of the runner's speed and the wind speed. (This is the situation for airplanes flying into a head wind or with a tail wind.)

This then gives us:

$$\text{Resultant speed out} = (v - u) \qquad (1)$$

$$\text{Resultant speed back} = (v + u) \qquad (2)$$

$$\text{Time out} = \frac{d}{v-u} \quad \text{Time back} = \frac{d}{v+u} \qquad (3)$$

Total running time in wind

$$= \frac{d}{v-u} + \frac{d}{v+u} \qquad (4)$$

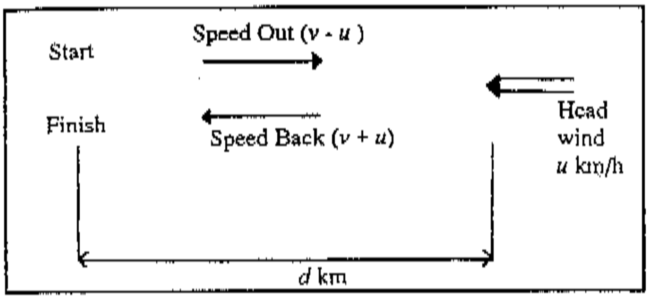


Fig. 1. An out-and-back race.

$$= \frac{2dv}{\left(v^2 - u^2\right)} \quad (5)$$

Total running time without wind

$$= \frac{2d}{v} \quad (6)$$

$$\text{Time difference } (\Delta T) = \frac{2dv}{(v - u)(v + u)} - \frac{2d}{v} \quad (7)$$

$$= \frac{2du^2}{v\left(v^2 - u^2\right)} \quad (8)$$

Conclusions from the First Model

To resolve the argument given in the introduction, let us examine the sign of ΔT. The numerator $(2du^2)$ is always positive, but so also is the denominator $(v^3 - u^2v)$, provided v is positive and $v > u$ (note that according to the model, if $u = v$, then no running would be possible). Thus, ΔT is always positive, provided $v > |u| > 0$ (u could be negative if it were blowing in the opposite direction). This means that it will always take longer running in the wind than without it, something that is well known from the classical analysis of the Michelson-Morley experiment.

Since the divisor $(v^3 - u^2v)$ of the ΔT fraction is a cubic polynomial in v, it increases rapidly with an increase in v, resulting in a corresponding decrease in ΔT. Slower runners are therefore affected far more by the wind than faster runners. So next time you run in the wind, don't slow down, but run as fast as you can to minimize the effects of the wind! (This advice is also logical, simply from the viewpoint that the longer you take to finish the run, the longer you are exposed to the effects of the wind.)

ΔT is a linear function of d and therefore for the same u and v, it increases directly proportional to d (e.g., under the same conditions ΔT for a marathon is twice that for a half marathon).

Since an increase in wind speed (u) increases the numerator of ΔT quadratically and decreases the denominator correspondingly, ΔT will increase dramatically with an increase in the wind speed.

Further Evaluation and Extension

How accurate is the model for certain given situations? Does it correspond at all to reality? If it does not, can we adapt it to conform better to reality?

These are important questions that we should always ask of mathematical models. It is important in science teaching not to let students confuse a given model with the reality it attempts to depict, and we should therefore constantly inculcate a critical attitude to modeling.

First of all, it is surely a crude assumption that the resultant velocities going out and coming back are merely the vector sums of the velocities involved. This would mean that when wind speed equals the running speed, no movement would be possible, and if it exceeds it, a runner would be blown back. Given that a slow runner runs about 10 km/h (6 min/km) this implies that he/she wouldn't be able to run at all even in a slight breeze (see Table I). From this table, we can also see that even world-class runners like Temane and Yawa (21 km/h : < 3 min/km) would no longer be able to run in a wind exceeding Beaufort scale number 3.

Thus, it is probably more accurate to assume that only a proportion of the wind speed affects one's running speed.

Table I. The Beaufort Scale.*

Scale number	Wind speed (km/h)	Description of wind	Indications on land
0	0–2	Calm	Smoke goes straight up
1	3–5	Light air	Smoke drifts
2	6–12	Slight breeze	Leaves rustle
3	13–20	Gentle breeze	Leaves and small twigs move
4	21–30	Moderate breeze	Small branches move; dust and paper fly
5	31–40	Fresh breeze	Ripples on water; small trees sway
6	41–52	Strong breeze	Large branches move
7	53–63	High wind	The trunks of trees bend; walking is difficult
8	64–77	Gale	Twigs are broken off
9	78–90	Strong gale	Chimneys and shingles are carried off
10	91–105	Full gale	Trees may be uprooted
11	106–125	Storm	Damage is widespread
12	over 125	Hurricane	Anything can happen

*Taken from Ref. 1, p. 93.

Table II. ΔT for a distance of 21 km. Data in top section for $k = 1$; data in bottom section for $k = 0.075$.

Running speed in km/h	Wind speed in km/h 1	2	5	10	20	30	40	50	70
10 (6 min/km)	1 min 16 s	5 min 15 s	42 min	—	—	—	—	—	—
15 (4 min/km)	23 s	1 min 31 s	10 min 30 s	1 h 7 min	—	—	—	—	—
20 (3 min/km)	10 s	38 s	4 min 12 s	21 min	—	—	—	—	—
10 (6 min/km)	0.4 s	2 s	11 s	43 s	2 min 54 s	6 min 43 s	12 min 28 s	20 min 37 s	47 min 57 s
15 (4 min/km)	0.1 s	0.5 s	3 s	13 s	51 s	1 min 56 s	3 min 30 s	5 min 36 s	11 min 44 s
20 (3 min/km)	—	0.2 s	1 s	5 s	21 s	49 s	1 min 47 s	2 min 18 s	4 min 40 s

Assuming this constant to be k, the running speeds out and back are, respectively, $(v - ku)$ and $(v + ku)$, giving:

$$\Delta T = \frac{2dk^2u^2}{v\left(v^2 - k^2u^2\right)} \tag{9}$$

Looking again at the Beaufort scale for indicating wind velocities in Table I, we might assume that a slow runner (10 km/h) would no longer be able to run when the wind speed exceeds 133 km/h, which is in the hurricane range. However, since $v - ku > 0$ for running to be possible, we can solve k from this equation by substituting $u = 133$ km/h and $v = 10$ km/h, giving $k = 0.075$. (Note that we are assuming k to be constant for all wind speeds and the same for all the runners.)

Some worked examples are now given in Table II for $k = 1$, as well as for $k = 0.075$. Compared with our initial model, our improved model certainly looks more realistic. Readers are now invited to further judge the reasonableness of the improved model's predictions against their own experience.

Although the second model is clearly an improvement on the first, a remaining problem is that the difference between the ΔT of a slow runner and that of a fast runner becomes absurdly high at high wind speeds. For instance, for $u = 100$ km/h, ΔT for a slow runner is 2 h 42 min, whereas for a fast runner it is only 10 min 18 s. Surely the difference cannot be that much! Furthermore, whereas running is no longer possible for the slow runner when $u = 133$ km/h, the model predicts that the fast runner (20 km/h) could continue to run until wind speeds of 267 km/h! By the time the wind reaches such speeds where the slow runner cannot move at all, surely neither would the fast runner be able to.

If we therefore assume that no runner could run faster than a certain maximum wind speed (u_{max}), k becomes a functional variable of v, in fact directly proportional to v (e.g., solving from $v - ku_{max} = 0$ we find $k = v/u_{max}$. Thus, for $u_{max} = 133$ km/h, we have $k = 0.0075\ v$, for $u_{max} = 150$ km/h, $k = 0.0067\ v$, and for $u_{max} = 200$ km/h, $k = 0.005\ v$.

By substituting $k = mv$ (where $m = 1/u_{max}$) into the ΔT formula, we obtain:

$$T = \frac{2dm^2v^2u^2}{v^3\left(1 - m^2u^2\right)} \tag{10}$$

$$= \frac{2dm^2u^2}{v\left(1 - m^2u^2\right)} \tag{11}$$

$$= \frac{2d\,(u/u_{max})^2}{v\left(1 - (u/u_{max})^2\right)} \tag{12}$$

Table III. ΔT for a distance of 21 km.

Running speed in km/h	Wind speed in km/h 1	2	5	10	20	30	40	50	70	
10	0.4 s	1.7 s	11 s	43 s	2 min 54 s	6 min 43 s	12 min 28 s	20 min 37 s	47 min 57 s	
15	0.3 s	1.1 s	7 s	29 s	1 min 56 s	4 min 29 s	8 min 18 s	13 min 45 s	31 min 58 s	$u_{max} = 133$ km/h
20	0.2 s	0.9 s	5 s	21 s	1 min 27 s	3 min 22 s	6 min 14 s	10 min 19 s	23 min 58 s	
10	0.3 s	1.3 s	8 s	34 s	2 min 17 s	5 min 15 s	9 min 39 s	15 min 45 s	35 min 5 s	
15	0.2 s	0.9 s	6 s	23 s	1 min 31 s	3 min 30 s	6 min 26 s	10 min 30 s	23 min 23 s	$u_{max} = 150$ km/h
20	0.1 s	0.7 s	4 s	17 s	1 min 8 s	2 min 37 s	4 min 49 s	7 min 53 s	17 min 32 s	
10	0.2 s	0.8 s	5 s	19 s	1 min 16 s	2 min 54 s	5 min 15 s	8 min 24 s	17 min 35 s	
15	0.1 s	0.5 s	3 s	13 s	51 s	1 min 56 s	3 min 30 s	5 min 36 s	11 min 44 s	$u_{max} = 200$ km/h
20	0.1 s	0.4 s	2 s	10 s	38 s	1 min 27 s	2 min 38 s	4 min 12 s	8 min 48 s	

Table IV. ΔT for a distance of over 21 km (with wind resistance proportional to relative speed squared).

Running speed in km/h	Wind speed in km/h: 1	2	5	10	20	30	40	50	70	
	8.0 s	9.3 s	18 s	50 s	3 min 2s	6 min 51 s	12 min 35 s	20 min 45 s	48 min 4 s	
10	7.9 s	8.7 s	15 s	36 s	2 min 4 s	4 min 36 s	8 min 26 s	13 min 52 s	32 min 5 s	u_{max} = 133 km/h
	7.8 s	8.4 s	13 s	29 s	1 min 35 s	3 min 29 s	6 min 21 s	10 min 26 s	24 min 6 s	
	7.9 s	8.9 s	16 s	42 s	2 min 26 s	5 min 26 s	9 min 53 s	16 min 3 s	35 min 39 s	
15	7.8 s	8.5 s	13 s	30 s	1 min 40 s	3 min 40 s	6 min 38 s	10 min 45 s	23 min 49 s	u_{max} = 150 km/h
	7.7 s	8.3 s	12 s	25 s	1 min 17 s	2 min 47 s	5 min 0 s	8 min 5 s	17 min 53 s	
	7.8 s	8.3 s	12 s	27 s	1 min 24 s	3 min 2 s	5 min 23 s	8 min 32 s	17 min 43 s	
20	7.7 s	8.1 s	11 s	20 s	58 s	2 min 4 s	3 min 38 s	5 min 44 s	11 min 51 s	u_{max} = 200 km/h
	7.7 s	8.0 s	10 s	17 s	46 s	1 min 35 s	2 min 45 s	4 min 20 s	8 min 55 s	

$$= \frac{2du^2}{v\left((u_{max})^2 - u^2\right)} \quad (13)$$

Whereas the denominator of the ΔT fraction was originally a cubic polynomial of v, it is now only a linear function of v. Consequently, under the same conditions, ΔT will still decrease for faster runners, but not by quite as great a margin as originally.

In Table III a comparison is given between the wind effects on slow, average, and fast runners for maximum wind speeds of 133, 150, and 200 km/h, respectively.

To further improve the model, we can allow for wind resistance by assuming that the running speed is decreased by an amount proportional to the relative speed squared, that is, assuming that the relative speeds going out against the wind and returning with it are respectively given by $(v - mvu) - r(v - mvu)^2$ and $(v + mvu) - r(v + mvu)^2$. In the same manner as before it then follows that

$$\Delta T = \quad (14)$$

$$\frac{2dm^2u^2 + 2drv - 6drvm^2u^2 - 2dr^2v^2 + 4dr^2m^2v^2u^2 - 2dr^2m^4v^2u^4}{v\left(1 - m^2u^2\right)\left(1 - rv + rmvu\right)\left(1 - rv - rmvu\right)}$$

In Table IV a comparison is now given between the wind effects on slow, average, and fast runners for $r = 0.0001$ and maximum wind speeds of 133, 150, and 200 km/h. Comparing Table IV with Table III, it seems that the further introduction of wind resistance in the model affects the lower wind speeds far more than the higher wind speeds. For example, while the relative increase in ΔT (for u_{max} = 133 km/h and compared with Table III) for a running speed of 10 km/h and a wind speed of 1 km/h is 1900 percent, it is only 0.24 percent in the case of v = 10 km/h and u = 70 km/h. Similarly, for u_{max} = 150 km/h, v = 10 km/h, and r = 0.001 the relative increase for u = 5 km/h is 963 percent, but for u = 50 km/h it is only 9.2 percent. Of course the higher the value of r the greater also the value of ΔT. For example, for r = 0.01, v = 10 km/h, and u_{max} = 150 km/h, ΔT for wind speeds of 1, 2, 5, 10, 50, and 70 km/h is, respectively, 14 min 0 s, 14 min 1 s, 14 min 9 s, 14 min 34 s, 29 min 57 s, and 49 min 34 s. From this it also appears that the higher the value of r, the smaller the variation in ΔT for lower wind speeds.

The accuracy of this model could now be verified experimentally, and the most appropriate constants selected for a specific range. Other similar examples for the classroom would be to examine the effects of a head wind (or a tail wind) on running performance for a one-way course, or on a track. Somewhat similarly, although slightly more complex, the situation can also be modeled for an uphill-downhill route.

Reference

1. J.R. Tannehill, "The Origin and Nature of the Earth's Winds," *The Book of Popular Science*, Vol. 2 (Grolier, New York, 1974), pp. 89–100.