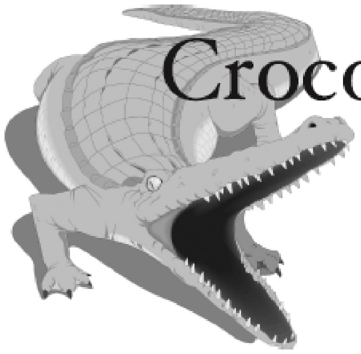
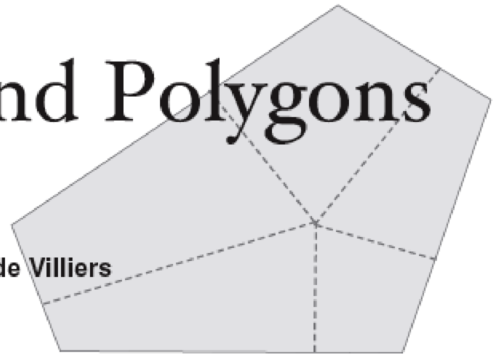
Published in *Mathematics in School*, March 2005, pp. 2-4.



Crocodiles and Polygons



by Michael de Villiers

Introduction

One of the constant challenges facing us as mathematics educators is to keep coming up with problems that genuinely surprise and intrigue our students (or better still and ideally, to stimulate our students to come up with variations themselves!). Such problems should provoke a need for explanation and be rich enough for further investigation to encourage students to branch off into different directions. It often helps to give some of these problems a "*real world*" flavour even though they aren't genuine applied mathematical problems. Further spice could be added in the form of a modest sprinkling with a touch of humour.

One possible example that I've found very useful in stimulating surprise among children as well as practising and prospective teachers is Viviani's theorem, which states that the sum of the distances from a point to the sides of an equilateral triangle is constant. In my *Rethinking Proof with Sketchpad* book (De Villiers, 2003), this problem is placed in a pseudo-real world context of a surfer stranded on an island in the shape of an equilateral triangle. The surfer wants to build a hut where the sum of the distances to the sides is a minimum as she surfs on each of the three beaches an equal amount of time.

By using dynamic geometry, students are inevitably very surprised by the unanticipated result that while the three distances change as they drag the point around, the sum nevertheless remains constant! So it wouldn't matter where the surfer builds her hut! Even though 14-year olds show little desire for further conviction having checked it themselves by dragging as reported in Mudaly & De Villiers (2000), their curiosity is usually sufficiently aroused to engage them meaningfully in a guided logical explanation.

Since the result is logically explained from the *equality of the sides*, it follows immediately that the basic argument is generalizable to any *equi-sided* polygon. Even

Published in *Mathematics in School*, March 2005, pp. 2-4. All rights reserved
by The Mathematical Association, http://www.m-a.org.uk

though considering a point outside an equilateral triangle (or any equi-sided polygon) is not sensible given the practical context of an island and the building of a hut, students are also asked to investigate what happens when the point falls outside. Investigation with *Sketchpad* or *Cabri* does not show a constant sum while the point is being dragged outside. It therefore comes as a further surprise and useful learning experience that the sum actually remains constant if we introduce the concept of "*directed distances*", allowing distances to become negative if they fall completely outside the polygon (see De Villiers, 2003, p. 149).

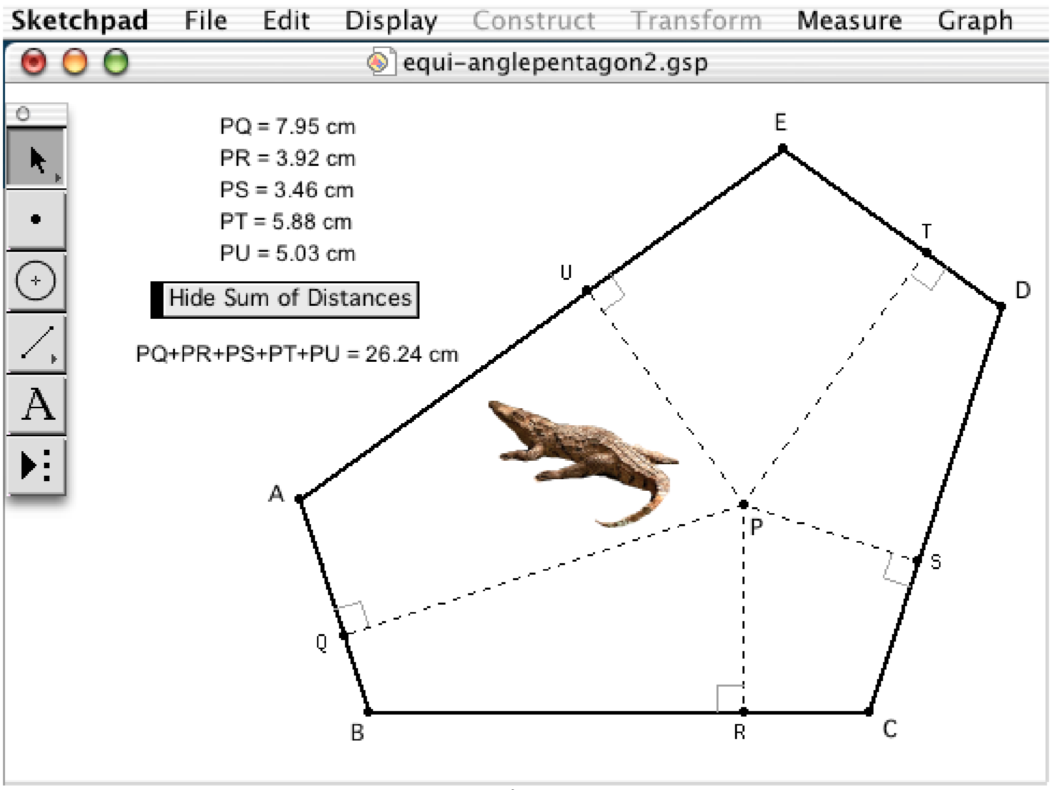


Figure 1

The Crocodile Problem

But the richness of the Viviani's theorem is hardly exhausted! Students can further explore (and explain) what happens when the triangle is not equilateral, as well why it works for a parallelogram which does not have equal sides (and then to consider its generalization to all (even-sided) polygons with opposite sides parallel). Another interesting generalization that students can explore with dynamic geometry is the following problem that students may find amusing:

> "*A mathematical crocodile (whatever that is!) in the Okavango delta lives in a swampy region in the shape of an equi-angled pentagon (see Figure 1). Since the crocodile captures prey an equal amount of times on each of the five banks, it (nicely non-gendered, not so?) wants to hide its captured prey where the sum of the five distances to the banks is a minimum. Where is this optimal point?*"

The reader is invited to perhaps first pause and explore the problem for a short while with a zipped Sketchpad sketch that can be downloaded directly from:
http://mysite.mweb.co.za/residents/profmd/crocodile.zip

Since the sides are not equal nor are there any opposite parallel sides, students do not at first expect that the sum of distances would still be constant, and are again surprised when it does not matter how they drag the point P. But why is it still true? How can we logically explain this result?

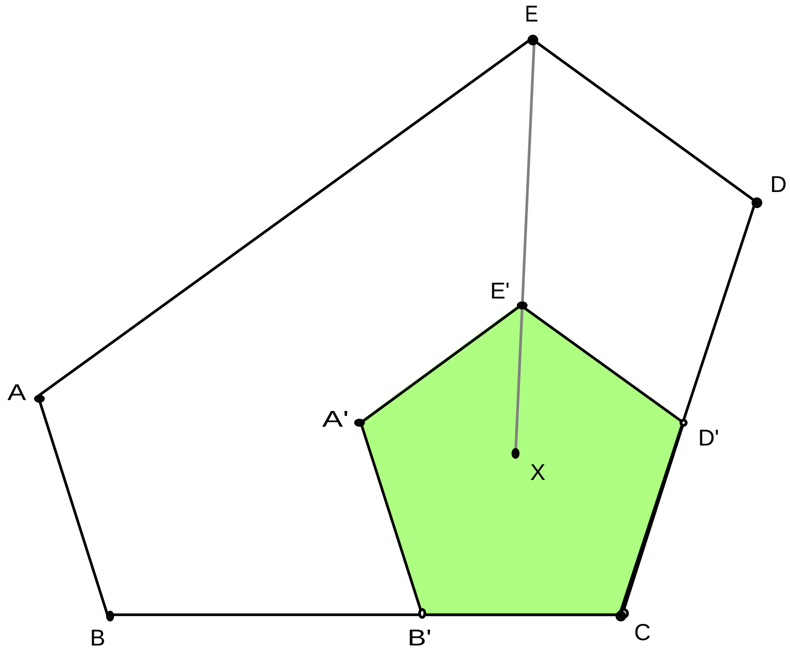


Figure 2

Logical Explanation (Proof)

Firstly note that since all the angles of $ABCDE$ are equal, each of its angles have to be the same as that of a regular pentagon, namely, 108°. By placing a regular pentagon in relation to an equi-angled pentagon as shown in Figure 2, it follows that since $\angle ABC = \angle A'B'C$ that AB // $A'B'$. In the same way, DE // $D'E'$, and therefore $\angle DEX = \angle D'E'X$ (where X lies on EE' extended). However, $\angle DEC = \angle D'E'C$; thus $\angle AEX = \angle A'E'X$ which means AE // $A'E'$. It follows that a regular pentagon $A'B'C'D'E'$ can always be placed inside an equi-angled $ABCDE$ so that their corresponding sides are parallel as shown in Figure 3.

Since the sum of distances to the sides of any regular polygon is constant (as all sides are equal), it follows that $\sum_{i=1}^{5} x_i$ is constant. However, the distances y_i between the corresponding parallel sides of the two pentagons are always constant, thus $\sum_{i=1}^{5} y_i$ is also constant. But the sum of these two constants must also be constant, and therefore completes the logical explanation and proof.

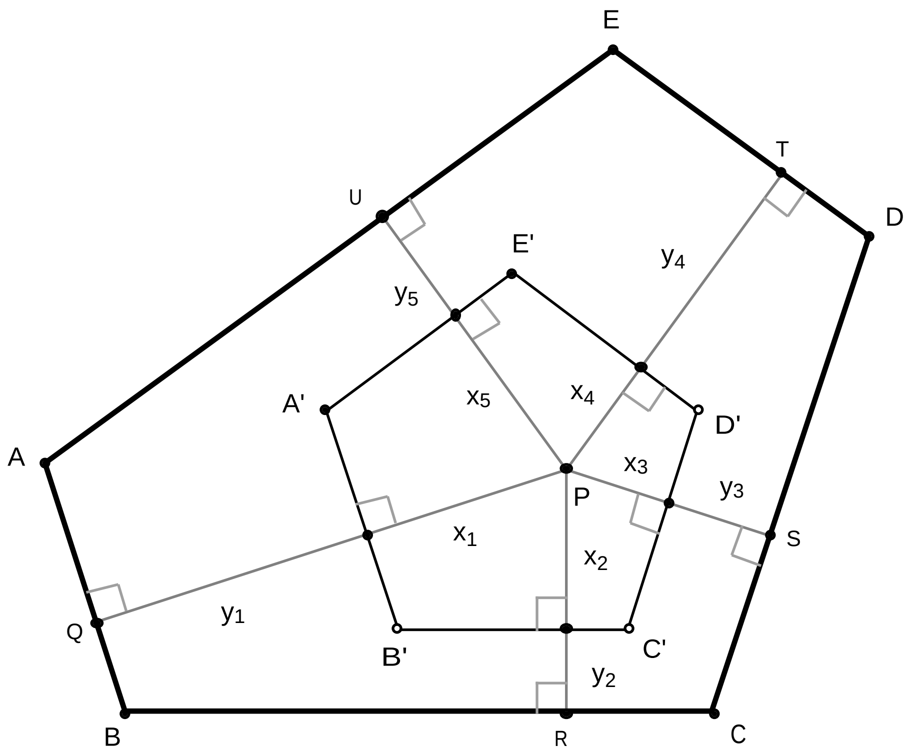


Figure 3

Note that directed distances are assumed above to cover cases when *P*, for example, lies outside *A'B'C'D'E'*. It is furthermore obvious from the proof that the result generalizes in exactly the same way to any *equi-angled* polygon.

Some Further Possibilities

One possibility is to ask students to think about a possible 3D analogue of the equilateral triangle version of Viviani's theorem. With a little prompting it usually doesn't take them long to realise that the result can be generalized to a tetrahedron with faces of equal area. The obvious next question is: for which types of polyhedra would this hold, and whether it can be generalized to even higher dimensions?

Students may also be interested to investigate whether Viviani's theorem also respectively holds in hyperbolic and elliptic geometry. For such an exploration the dynamic geometry software *Cinderella* is ideally suited (see Figure 4). Visit the website http://www.cinderella.de for more information and to download a demo.

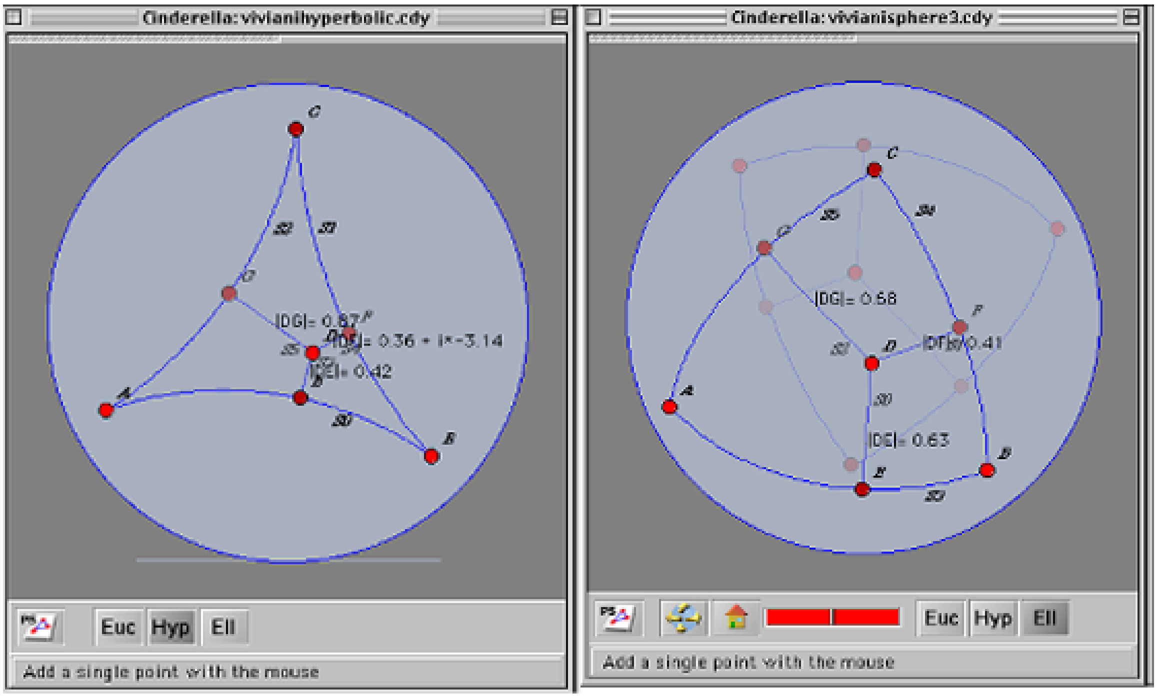


Figure 4

Acknowledgement

I am thankful to John Silvester from the Department of Mathematics, King's College, London, for kindly steering me in the direction of the simple proof and explanation given above.

References

De Villiers, M. (2003). *Rethinking Proof with Sketchpad*, Key Curriculum Press, Emeryville: USA. (A sample of zipped *Sketchpad* sketches from a few activities from the book can be directly downloaded from: http://mysite.mweb.co.za/residents/profmd/rethink2.zip)

Mudaly, V. & De Villiers, M. (2000). Learners' Needs for conviction and Explanation within the Context of Dynamic Geometry. *Pythagoras*, August, No. 52, pp. 20-23. (A PDF copy of this article can be downloaded directly from: http://mysite.mweb.co.za/residents/profmd/vim.pdf)

Published in *Mathematics in School*, March 2005, pp. 2-4.

Keywords: Dynamic geometry, logical explanation, proof, Viviani, generalization.

Author: Michael de Villiers, Mathematics Education, University of KwaZulu-Natal,
Ashwood 3605, South Africa. E-mail: profmd@mweb.co.za;
Homepage: http://mysite.mweb.co.za/residents/profmd/homepage.html