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Mathematical Modeling and Proof

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In this paper, it is argued that modeling could be seen as the first stage of the proving process. An experiment conducted with grade 10 (15 year old) learners in a small suburb in Durban will be discussed. The experiment showed that as a result of the modeling process, learners felt the need to know why the result was true, not because they doubted whether the result was true, but due to the innate desire for an explanation. A lot of work on similar topics have been done elsewhere in the world, but not much has been done in South Africa. Furthermore, the experiment was conducted using Sketchpad as a mediating tool. This in itself was a difficult task because our learners have not been really exposed to dynamic geometry environments.

Introduction

Proof is often only seen as a means of simply verifying the truth of mathematical statements. It is a perception that is commonly propagated in mathematics classrooms, where teachers focus on the verifying of mathematical truths that are being investigated. Seldom is there a link established between mathematical modeling of real-world problems and proving. This is also the finding of Hodgson and Riley (2001: 724), who stated that that "*proof and real-world problem solving are typically considered to be separate and distinct endeavors*". It has always been difficult to gauge the relationship between real-world problem solving and proof, yet the clear value of real-world problem solving in the process of proving cannot be underestimated.

Hodgson and Riley (2001 : 724) further state that "*our experience has been that real-world problems supply an important ingredient that seems to be missing from typical classroom instruction on proof. As such, real-world problems may actually be one of the most effective contexts for introducing and eliciting proof. Real-world problems are commonly* *used as vehicles to introduce or deepen students' understanding of mathematical concepts and relationships*".

Hodgson's and Riley's argument that real-world problems could be the basis for mathematical proof stems from one step in the modeling process, namely, the testing of the solution. They believe that it is essential for the students to ask "*why is the statement true*?" after they have arrived at the solution. In their experiment the students found that their solution was incorrect and this initiated the desire for an explanation. It is my contention that had they not gone through the process of modeling, it is unlikely that they would have wanted an explanation.

Similarly, Klaoudatos and Papastavridis (forthcoming), discuss a teaching experiment based on *Context Oriented Teaching* (COT). According to Klaoudatos and Papastavridis (page 1), COT is “*a model based on a problem solving framework and on the selection of the appropriate task context*”. They observed that COT provided the student, who had little understanding of the mathematics involved in solving a particular problem, a starting point and a sense of direction (p. 4). Essentially, they conclude that starting with a Context Oriented Question (which is an adapted real world question), the learners use Context Oriented Heuristics to develop Context Oriented Concepts. Context Oriented Conjectures are formulated, which leads to Context oriented Proofs. Despite framing their arguments within the idea of contexts, they still show that the modeling activity serves as a starting point for this proof.

A further significant argument for a direct link between modeling and proof is made by Blum (1998) when he stated that applications in mathematics (solving of real world problems) provide contexts for what he refers to as *reality-related proofs* (p. 1). He clearly points out that “***formal** proofs are mostly the final stage in a genetic development – historically as well as epistemologically as well as psychologically*”.

However, Klaoudatos and Papastavridis as well as Blum discuss proving in relation to modeling in teaching situations, which have been explicitly designed so as to connect the two. The question still remains whether the modeling activity will still serve as a starting point for proof if the specific modeling activity was not constructed with the intention of arriving at a proof. Furthermore, it is relevant to ask whether the experiences gained from the European contexts will be similar in South African classrooms, with different traditions, and teacher-student relationships and numbers. Below, both questions will be addressed.

The process of modeling and its relationship to proving

In order to understand the relationship between proof and modeling, one needs to have some knowledge of the processes involved in modeling. Modeling is not an easy task. It often involves a process of creating a miniature problem, which is analogous to the larger problem, but enabling the modeler to draw exact conclusions, which can be extrapolated to the original real life problem. Although the model attempts to simulate the original problem it cannot truly replicate all the constraints that might be imposed by the problem itself. It is similar to children who simulate adulthood by playing mothers and fathers, play with dolls, which are non-living entities, and cannot move like the way human babies do.

Modeling usually begins with a real life situation, which may be relatively controlled (for example, determining the profit of a manufacturing company), or sometimes in environments in which the modeler cannot control all the conditions (for example determining the population increase of fish in a river). In all cases the modeler is hoping to predict future behavior of the system under prevailing conditions.

De Villiers (1993: 3) describes three different categories of model application namely, *direct application* ("*immediate recognition of a model to be used*"), *analogical application* ("*development of a model that is similar to an existing model*") and *creative application* ("*a completely new model is created using new techniques and concepts*"). The experiment described below is entrenched in the latter category, where modeling is used as a teaching approach for developing and introducing new content.

In evaluating the model and its results the modeler begins to ask ***why*** does the result hold true? (Or even *why* does the result *not* hold true?). It is this question that clearly defines the relationship between modeling and proving. Asking ***why*** indicates a desire to seek some sort of an explanation. It is also clear that the question is not ***whether*** the result is true because the modeler has already convinced him/herself that the result is true during the interpretative stage. Once convinced the modeler develops a certain curiosity as to why the result is true and possible under such conditions. In attempting to answer this question the modeler begins to develop an explanation for the observed result and hence establish a proof valuable in increasing the understanding of the problem.

In a modeling experiment conducted with Grade 10 students, to teach concepts such as perpendicular bisectors, equi-distance, and concurrency, it was found that these students displayed a definite desire for a proof (as explanation). The investigator (Mudaly, 2004) initially envisaged that the actual proof of their discovery would not be necessary to develop and pursue, because the aim of the investigation simply was to determine whether *Sketchpad* could be used as a modeling tool in developing new concepts (such as concurrency and perpendicular bisectors). Later, I felt that it was useful and necessary to investigate whether learners could actually be guided to a simple proof based on their modeling activity. The activity and proof was based on materials developed by De Villiers (1999:32). It must be emphasized that it was not the intention during this experiment to teach modeling as much as to develop the mathematical concepts mentioned above. Modeling was therefore not the goal but intended as a vehicle.

The following real world problem was given to the students, contextualized within the South African rural background, which learners easily related to:

> *In a developing country like South Africa, there are many remote villages where people do not have access to safe, clean water and are dependent on nearby streams or rivers for their water supply. With the recent outbreak of cholera in these areas, untreated water from these streams and rivers has become dangerous for human consumption. Suppose you were asked to determine the site for a water reservoir and purification plant so that it would be the same distance away from four remote villages. Where would you recommend the building of this plant?*

The students were presented with diagrams already constructed using *Sketchpad* as a mediating tool. The diagrams were constructed in such a way that it accurately modeled the above situation with a dynamic sketch. The students first went through the process of working with this given model and discovered that for these four villages there existed a *unique equi-distant point* (where the perpendicular bisectors of the sides quadrilateral were concurrent), but that this was not generally true for any quadrilateral (only for those that were cyclic). For non-cyclic quadrilaterals, no unique equi-distant position existed for building such a reservoir!

They were then asked a similar question related to three villages. They readily conjectured that perpendicular bisectors would be concurrent for only those triangles, which were cyclic. They were enormously surprised when they used the drag function of *Sketchpad* to discover that the perpendicular bisectors of ***all*** triangles were concurrent (unlike it had been the case for quadrilaterals).

When asked if they wanted to know why this was true, all students gave the impression that they wanted an *explanation* in order to extract some understanding from it.

Clearly the proof was not required because they doubted the result, but because it seemed that they felt that it would satisfy some innate curiosity around the *reason* for the result.

An example of a student's proving process and strength of conviction

One example of the proving process is sufficient to convey the gist of what transpired after the students felt a need for an explanation.

TEACHER	Look at this triangle on the screen again. Construct the perpendicular bisector of any side.
DESIGAN	Can I do it for AB?
TEACHER	Yes. (*after the construction*) Desigan, what can you tell me about all the points on this perpendicular bisector?
DESIGAN	It is equidistant from A and B.
TEACHER	What is equidistant? (*trying to ascertain for sure what the 'it' was*)
DESIGAN	All the points on this line (*pointing to the perpendicular bisector*).
TEACHER	What does that really mean to you?
DESIGAN	If you measure the distance from any point on this line to this A and B, the distance will be the same.

In this segment the researcher was simply attempting to get the student to recall the concepts of perpendicular bisector and equidistance. In a way, it was also a means of determining whether the student actually understood and remembered what they had done earlier in the interview.

TEACHER	Now construct any other perpendicular bisector.
DESIGAN	(*constructing*)
TEACHER	What can you tell about the points on this line now?
DESIGAN	All the points are the same distance away from B and C.
TEACHER	Now look at this point of intersection. What can you say about this point in particular?
DESIGAN	Eh … eh…
TEACHER	Think carefully about the point.
DESIGAN	That point there is the same distance away from A and B and, B and C.

TEACHER	A and B and, B and C?
DESIGAN	Yes, it is the same distance away from A, B and C.
TEACHER	Are you sure?
DESIGAN	It lies on this line so it must be equidistant from A and B and it lies on that line so it must be equidistant from A and C (*note the error*).
TEACHER	If it lies on that line would it be equidistant from A and C?
DESIGAN	No B and C.
TEACHER	So are you sure that this point of intersection is the same distance away from A, B and C?
DESIGAN	Yes.

Initially, it seemed that this student realized that the point of intersection was equidistant in a fragmented way. In other words, he could see that the point of intersection was equidistant from A and B and B and C separately. He did not instantaneously see the connection between all of the vertices to the point of intersection.

TEACHER	This you have to think about very carefully. What can you say about the perpendicular bisector of AC?
DESIGAN	All the points will be equidistant from A and C.
TEACHER	Yes, that is correct. But look at the other perpendicular bisectors.
DESIGAN	(*silence for a while*) ….Oh yes, it must pass through the point where these two lines meet (*pointing to the perpendicular bisectors*).
TEACHER	Really?
DESIGAN	Yes because if all the points on this perpendicular bisector of AC are the same distances away…then … then this point of intersection is also the same distance away .. then…
TEACHER	Yes?
DESIGAN	Then the line must pass through the point of intersection.
TEACHER	Are you absolutely sure that this would happen?
DESIGAN	Yes, I'm positive.
TEACHER	Do you want to see whether that is true?
DESIGAN	Yes.
TEACHER	Construct the perpendicular bisector of AC then.

DESIGAN	(*after constructing*) This is so easy.
TEACHER	Was it really that easy?
DESIGAN	I didn't take so long to get it right!

Eventually, when he realized that there was a connection between the three vertices and the point of intersection, the rest of the explanation became simple. It was clear that because these learners had initially worked, in the modeling process, with the concept of equidistance, the actual proof became easy to understand. This is supported by the fact that the student felt that this explanation was quite easy, and furthermore, he felt that he alone had got it correct. This clearly seems attributable to the high level of conviction that could be achieved using dynamic geometry software such as *Sketchpad* as a mediating tool.

It must also be pointed out that during the modeling process itself, the student was encouraged to find the link between the real world situation and the modeling activity itself. When the students were asked what this result meant in terms of the three villages, some of the responses were as follows:

DESIGAN	You can join the three villages and then find the perpendicular bisectors. Where they meet is the important point for us to use.
TEACHER	Do you think that it is easy to just join these villages and find the perpendicular bisectors?
DESIGAN	I don't think that its easy … I'm sure they can draw it on a page first and then do an exact drawing … or even use this programme to get the exact position.
TEACHER	Do you think that this would be easy to do in real life?
DESIGAN	I don't know… we must consider a lot of factors … like we discussed in the beginning.

VISCHALAN	If the villages are situated like this triangle then all you have to do is join the villages, find the midpoints between them and construct the perpendicular bisectors. The point of concurrency will be the most suitable point.
TEACHER	Do you think that it is easy to just join these villages and find the

	perpendicular bisectors?
VISCHALAN	Yes … you can use a map of the area.
TEACHER	Do you think that this would be easy to do in real life?
VISCHALAN	My uncle told me that you can use GPS (*Global Positioning System*) to find any point you want. I think the government has pictures of every part of the country.

It was evident from some of their responses that these students were quite capable of transposing real world problems into mathematical systems and returning to the real world as they see it.

Conclusion

This investigation, despite not intending to establish a proof for a result, unwittingly became an excellent springboard for the eliciting of logical reasoning from the students. The question might not have been an actual real-world problem, but it allowed for the student to have a partial interaction with the modeling process. It was quite clear that going through the modeling process evoked from the students a strong desire for a proof in order to increase their understanding of why the result was true. An argument can be made that this was simply the result of their high levels of conviction due to the effectiveness of *Sketchpad*, but the evidence presented by the students also indicated that the modeling process and relation to the real world also served as a basis and a stimulus for the proof of the concurrency of perpendicular bisectors of triangles. Concurrency and cyclic quadrilaterals are generally only done in the Grades 11 or 12 in South Africa, though the students would have started to interact with proofs of theorems in Grade 10. It was significant that these Grade 10 learners felt a strong desire to seek an explanation for the reasons for concurrency. Moreover, the high levels of conviction established during the modeling process probably served to form the impetus for the final proof. Lastly, it should be noted that the proof that was effective as an explanation (and based on the idea of equi-distance) differed from the traditional South African textbook proof based on congruency (i.e. following a traditional Euclidean approach).

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