From *Teaching & Learning Mathematics*, No. 4, Feb 2007, pp. 25-27, a journal of the Association of Mathematics Education (AMESA). http://academic.sun.ac.za/mathed/AMESA/

Further reflection on a SA Mathematics Olympiad Problem

MICHAEL DE VILLIERS

Visiting Professor, Dept. of Mathematics & Statistics, Kennesaw State University, USA

(On sabbatical from UKZN-Edgewood)

e-mail: *profmd@mweb.co.za*

http://mzone.mweb.co.za/residents/profmd/homepage.html

Every year the SA Mathematics Olympiad produces a valuable resource of new, innovative mathematical problems, and manages to stimulate interest in problem solving among learners, teachers and parents. Apart from being an essential resource for preparation for SAMO, a number of problems can also be used as further investigations by teachers. Past papers with solutions from 1997 to 2006 are freely available at http://ridcully.up.ac.za/samo/questions.html

The Senior Second Round of the 2006 Harmony SA Mathematics Olympiad contained an interesting problem in Question 20. For example, consider Figure 1. If $AE = 3$, $DE = 5$, and $CE = 7$, then BF equals

(a) 3.6 (b) 4 (c) 4.2 (d) 4.5 (e) 5

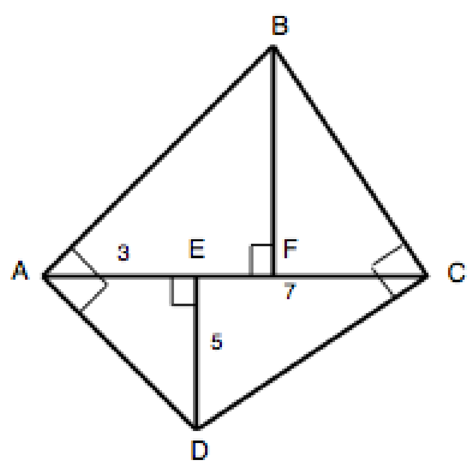

Figure 1

Before reading further the reader is encouraged to perhaps stop and first try to solve this problem **before** reading further.

From *Teaching & Learning Mathematics*, No. 4, Feb 2007, pp. 25-27, a journal of the Association of Mathematics Education (AMESA). http://academic.sun.ac.za/mathed/AMESA/

The following solution is provided: **Answer C**. By looking at the angles we see that triangles AED and BFA are similar, as are triangles CFB and DEC. Therefore $\frac{BF}{AF} = \frac{AE}{DE} = \frac{3}{5}$ and $\frac{BF}{CF} = \frac{CE}{DE} = \frac{7}{5}$, so $\frac{CF}{AF} = \frac{3}{7}$. But $CF + AF = CE + AE = 10$, so $CF = 3$ and $AF = 7$. Therefore $BF = \frac{BF}{AF} \times AF = \frac{3}{5} \times 7 = 4.2$.

Problem 20 is intended to be the toughest problem on the Second Round, but should also not be too hard. Problem 20 involves similarity, which is usually fairly tough for learners. It is therefore perhaps not surprising that only 22% got it right, and turned out to be one of the toughest problems in the Second Round. So for once, the Senior SAMO committee, which includes myself, got the level more or less right!

However, the problem and solution masks an interesting underlying theorem. The observant reader may have noticed that $CF = 3 = AE$ and $AF = 7 = CE$, and perhaps wondered whether this was mere coincidence of the choice of numbers. Surprisingly though, this would always be the case for any quadrilateral $ABCD$ with opposite right angles at A and C!

More precisely, the underlying theorem can be formulated as follows: Given any quadrilateral $ABCD$ with right angles at A and C, and the perpendiculars from B and D to the diagonal AC meet it at F and E respectively, then $AE = CF$ (and $CE = AF$). The result follows directly from generalizing the above solution by letting, say, $AE = p$, $DE = q$, and $CE = r$, and is left as an exercise for the reader.

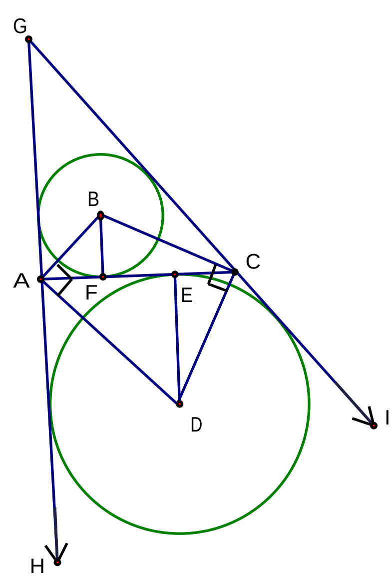

Figure 2

From *Teaching & Learning Mathematics*, No. 4, Feb 2007, pp. 25-27, a journal of the Association of Mathematics Education (AMESA). http://academic.sun.ac.za/mathed/AMESA/

But there is even more intrigue to the underlying theorem! Since there are right angles at *F* and *E*, we can draw two circles centred at *B* and *D* respectively as shown in Figure 2. Assume the two circles are of different size with circle *B* smaller than circle *D*. Then draw tangents from *A* and *C* to circles *B* and *D* respectively. If $\angle BAC = x$, then angles *GAB*, *DAC* and *DAH* are respectively *x*, $90° - x$ and $90° - x$, and it follows that *GAH* is a straight line. Similarly, *GCI* is a straight line. So now we have the following surprising result which does not appear to be well-known (see Note 4): Given any triangle *GAC*, then if *F* and *E* are the respective tangent points of the triangle's incircle and excircle to *AC*, then *AE* = *CF* (and *CE* = *AF*). Similarly, the result follows for the other two excircles.

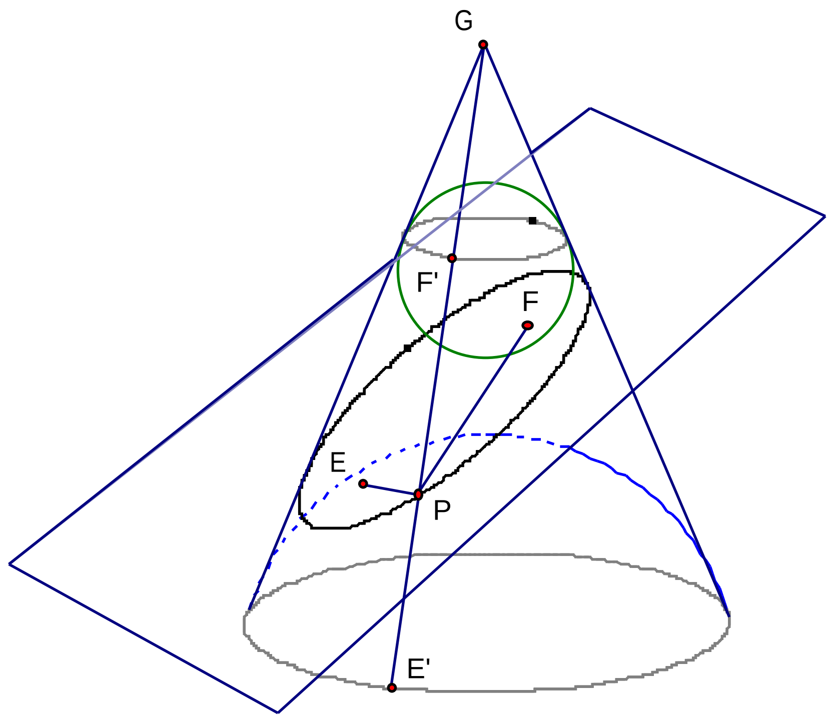

Figure 3

Viewed this way, this result can also be seen as just a special case of a beautiful 3D result discovered by Germinal Dandelin (1794 – 1847). His theorem states that if we inscribe two spheres into a circular cone to respectively touch the slicing plane on both sides, then not only is the intersection of the plane with the cone an ellipse (already known to the ancient Greeks), but the two spheres respectively touch the plane at the focal points of the ellipse (see Figure 3). Dandelin's theorem is surprisingly easy to prove.

Proof: Let the points where the top and bottom spheres respectively touch the slicing plane be F and E. Further let P be an intersection of the plane with an edge of the cone, F' and E' be the points where the line PG respectively touch the upper and lower spheres. Then since P, F and F' lie on a plane, we have that $PF = PF'$ (tangents from a point to a circle are equal). Similarly, $PE = PE'$. Therefore, $PF + PE = PF' + PE' = F'E'$, which is constant. Therefore, by definition of an ellipse, the intersection of the plane with the cone is an ellipse, and its focal points are E and F.

Figure 2 is clearly obtained when the cone is viewed directly from the side, where A and C are the points where the main axis of the ellipse intersects the sides of the cone. The result that $AE = CF$ (and $CE = AF$) then follows immediately from the symmetry of the ellipse.

This is therefore another good illustrative example, as will be discussed further in De Villiers (In press), of where a 2D result can be proved much more easily by considering it as a special case of a 3D result!

Notes

1) A quadrilateral with a pair of opposite right angles is called a *right quadrilateral* in De Villiers (1996), and apart from being cyclic, also has the interesting property that if the right angles are at A and C, then $AB^2 + AD^2 = CB^2 + CD^2$.

2) SAMO is open to all high school learners and information about participation is available at http://ridcully.up.ac.za/samo/ or from: The Secretary, SAMO, Private Bag X173 , Pretoria 0001. Tel/Fax: 012-3201950; E-mail: ellie@samf.ac.za

3) Dynamic Geometry (*Sketchpad 4*) sketches in zipped format (Winzip) of the results discussed here can be downloaded directly from:
http://mysite.mweb.co.za/residents/profmd/samodandelin.zip
(If not in possession of a copy of *Sketchpad 4*, these sketches can be viewed with a free demo version of *Sketchpad 4* that can be downloaded from:
http://www.keypress.com/sketchpad/sketchdemo.html)

4) Prof. Nic Heideman from the Mathematics Department at UCT, and Chair of SAMO, assures me that the result mentioned in Figure 2, and many similar ones, are quite well known in the International Mathematical Olympiad (IMO) training

context of the South African team.

These problems are all related to what has for about a dozen years now been referred to as the "Ravi substitution". Ravi used it with devastating effect, in order to simplify an inequality, in a top competition in the early 90's and he was one of the founders of the Canadian problem journal *Mathematical Mayhem*.

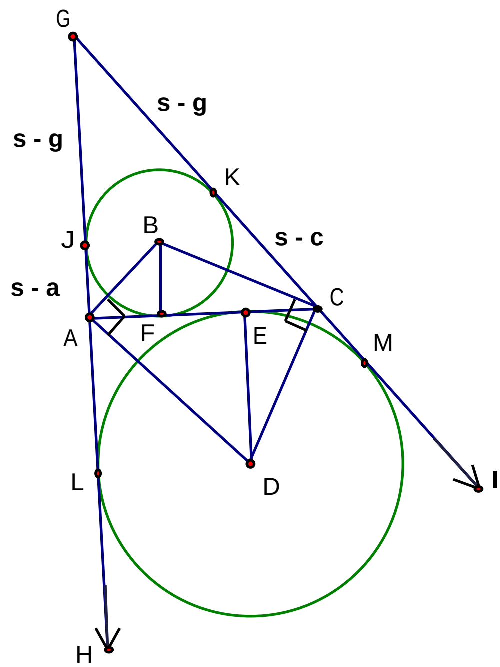

Figure 4

For example, consider Figure 4, where J, on GA, and K, on GC, are the points of contact with the inscribed circle B, and let L, on AH, and M, on CI, be the points of contact with the escribed circle D. Let $2s = g + a + c$ be the perimeter of triangle GAC, where $GC = a$, etc. Then by Ravi's substitution (left to the reader to verify) $GJ = GK = s - g$, $AJ = AF = s – a$, and $CF = CK = s – c$. Note that $GL = GA + AL = GA + AE$ and $GM = GC + CM = GC + CE$. Therefore, $GL + GM = GA + AE + GC + CE = 2s$. But $GL = GM$, hence $GL = GM = s$. Since $AF = s - a$, $AE = AL = s - (s - g) - (s - a) = (a + g) - s = (2s – c) - s = s - c = CF$.

References

De Villiers, M. (1996). *Some Adventures in Euclidean Geometry*. University of Durban-Westville (now University of KwaZulu-Natal).

De Villiers, M. (In press). Problem solving and proving via generalization. *Learning and Teaching Mathematics.*