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Solutions to Reader Investigations, 2005

Michael de Villiers, UKZN (Edgewood Campus); profmd@mweb.co.za

1. Since $\left(a^2 - b^2\right) = 20(a - b)$, using the difference between squares gives us $a + b = 20$. By investigation, the possible pairs of positive integers are (20, 0); (19, 1); (18, 2); (17, 3); (16, 4); (15, 5); (14, 6); (13, 7); (12, 8); (11, 9); (10, 10). So there are 10 more such pairs.

2. A general secret to analysing games of strategy is to work backwards from the end. The place to start is when (if!) the game gets down to just two pirates, P1 and P2. The fiercest pirate is P2, and (if the game ever gets this far) his optimal decision is obvious: propose 100 pieces for himself and none for P1. His own vote is 50% of the total, so it wins. Now add in pirate P3. Pirate P1 knows - and P3 knows that he knows - that if P3's proposal is voted down, P1 will get nothing. So P1 will therefore vote in favour of anything that P3 proposes, provided it yield him more than nothing. P3 therefore uses as little as possible of the gold to bribe P1, leading to the following proposal: 99 to P3, 0 to P2, and 1 to P1. Let's write it out like this:

P1	P2	P3
1	0	99

The thought processes of P4 are similar. He needs 50% of the vote, so again he needs to bring exactly one other pirate in board. The minimum bribe he can use is one gold piece, and he can offer this to P2 since P2 will get nothing if P4's proposal fails and P3's is voted on. So the proposed allocation becomes:

P1	P2	P3	P4
0	1	0	99

The thought processes of P5 are slightly more subtle. He needs to bribe *two* pirates. The minimum bribe he can use is two gold pieces, and the unique way he can succeed with this number is to propose the allocation:

P1	P2	P3	P4	P5
1	0	1	0	98

The analysis proceeds in the same manner, with each proposal being uniquely prescribed by giving the proposer the maximum possible, subject to ensuring a favourable vote, until we get to the tenth pirate and find the allocation, and solution of the problem:

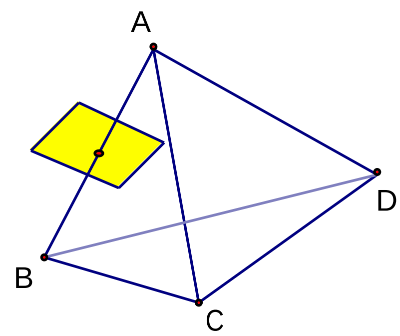
P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
0	1	0	1	0	1	0	1	0	96

Further investigation: What happens if the number of pirates are increased to 100, 200, and 500, while the number of gold coins remains fixed at 100?

3. The following is slightly adapted from Zeeman, C. (2005). *Three-dimensional theorems for schools*. Leicester: The Mathematical Association.

Let us start with following useful assumption (stated without proof) for 3D:
Three (non-parallel) planes meet in a point.

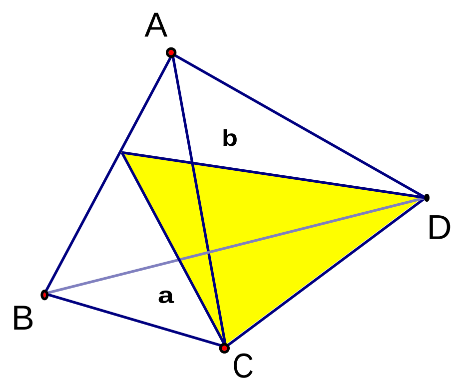
Definition 1: The *edge bisector* is the generalisation to 3D of the concept of a perpendicular bisector. For example, the edge bisector of *AB* is the plane through the midpoint of *AB*, and perpendicular to it. Or equivalently, it is the set of points equidistant from *A* and *B.*



Theorem 1: The six edge bisectors of a tetrahedron all meet at the circumcentre.

Proof: Let *S* be the intersection of the three edge bisectors of *AB*, *BC*, *CD*. Therefore *SA* = *SB* = *SC* = *SD*. Therefore *S* must lie on all 6 edge bisectors, and the sphere, centre *S* and radius *SA*, goes through all four vertices.

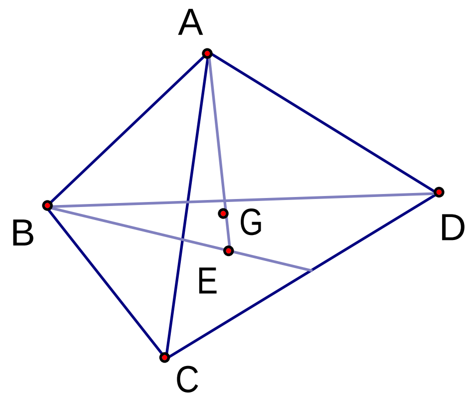
Definition 2: Let *a, b*, *c*, *d* denote the four faces of a tetrahedron *ABCD.* The *edge angle bisector* of *ab* is the plane through *CD* bisecting the angle between the faces *a*, *b*. It is therefore also the set of points equidistant from *a* and *b*.



Theorem 2: The six edge angle bisectors of a tetrahedron all meet at the incentre.

Proof: Let *I* be the intersection of the three edge angle bisectors *ab, bc, cd.* Then *I* is equidistant from all four faces, and is therefore the centre of the insphere touching all four faces.

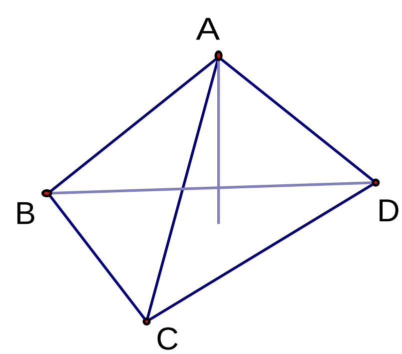
Definition 3: A median of a tetrahedron is the join of a vertex to the centroid of the opposite face.



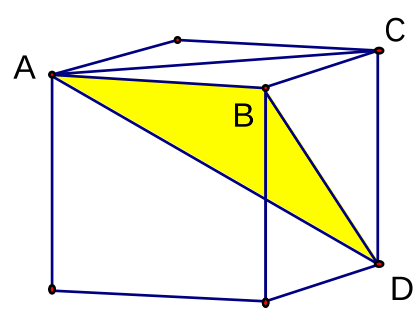
Theorem 3: The four medians of a tetrahedron meet at the centre of mass G.

Proof: Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be coordinate vectors of A, B, C, D. Then $\mathbf{e} = \frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d})$ is the centroid E of BCD. Let G be the point $\mathbf{g} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$. Then G lies on the median AE because $\mathbf{g} = \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{e}$. Similarly G lies on all four medians.

To verify that G is the centre of mass of the tetrahedron, note that the line containing BE divides triangle BCD into two triangles of equal area. Therefore the plane containing ABE divides the tetrahedron into two subtetrahedron of equal volume (they also have the same height). Therefore the centre of mass lies in this plane, and similarly in the plane containing ACE, and hence on AE. Similarly the centre of mass lies on all the medians, and hence is G.



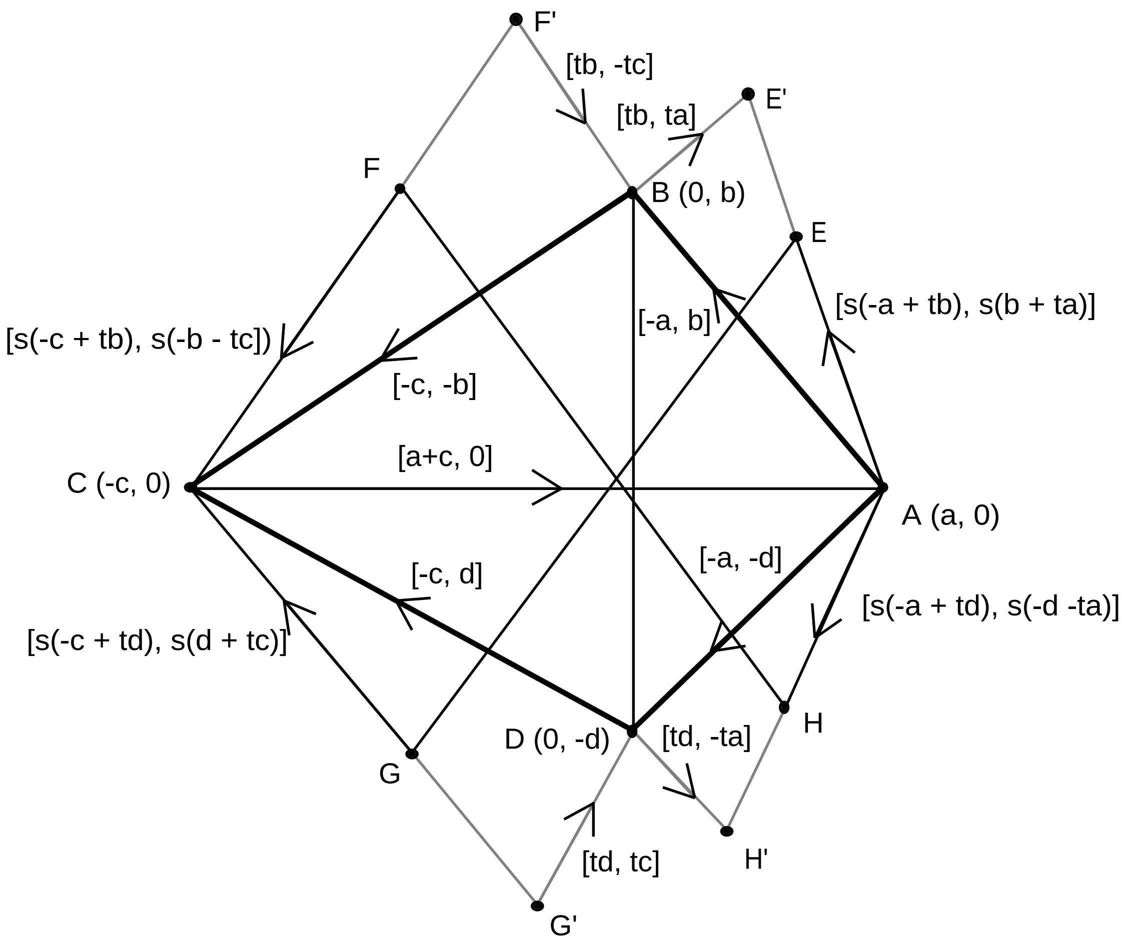
Definition 4: The altitude of a tetrahedron through A is the line perpendicular to BCD.



In general the four altitudes of a tetrahedron do not meet. It suffices to give a counter-example. Consider Dehn's tetrahedron $ABCD$ inscribed in a cube as shown. The altitudes through A, D are AB, CD which do not meet.

4. This result was experimentally discovered with *Sketchpad* by Michael de Villiers in 2004, though it is not known whether it is original. Two different, elegant proofs by Michael Fox from Leamington Spa, Warwickshire, UK, e-mail: mdfox@foxleam.freeserve.co.uk ; are given below.

(a) Proof by vectors



Given: Quadrilateral *ABCD* has perpendicular diagonals and triangles *AEB*, *AHD*, *CFB*, *CGD* are similar.

To prove: The lengths *FH*, *EG* are equal.

Proof: We use vector displacements, taking *A* as (a, 0), *B* as (0, b), *C* as (-c, 0), *D* as (0, -d).

Triangle *BE'A* is right angled at *B*, with *E'* lying on *AE*.

Displacement **AB** is [-a, b], so, if $t = \tan(\angle BAE)$, then **BE'** = [ta, tb], and **AE'** = **AB** + **BE'** = [-a + tb, b + ta].

If $\frac{AE}{AE'} = s$, then **AE'** = s **AE** = [s (-a + tb), s (b + ta)].

The other vector displacements follow similarly.

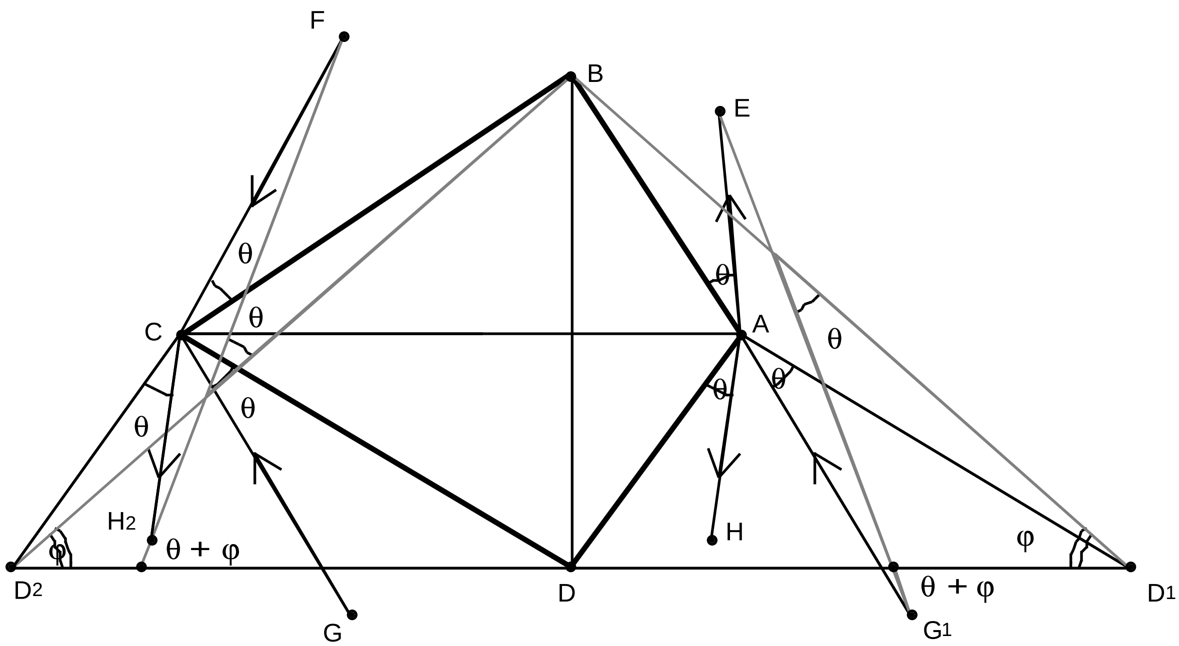
Then **FH** = **FC** + **CA** + **AH** = [$a + c$, 0] + s [$-a - c + t(b + d)$, $-b - d - t(a + c)$];

and **GE** = **GC** + **CA** + **AE** = [$a + c$, 0] + s [$-a - c + t(b + d)$, $b + d + t(a + c)$].

Since the x-components in each expression are equal, and the y-components are equal and opposite, the displacements have equal magnitudes, i.e. the lengths *FH*, *GE* are equal.

We also see that these lines are equally inclined to the diagonals *AC*, *BD*.

(b) *Proof vectors with transformations*



Proof: Since **FH** = **FC** + **CA** + **AH**, and **GE** = **GC** + **CA** + **AE**, both containing **CA**, the result would follow if the vectors **FC** + **AH**, **GC** + **AE** had equal magnitudes and were equally inclined to **CA**.

Translate *DC*, *GD* by vector **CA**. Their images are D_1A, G_1A, thus **GC** + **AE** = $\mathbf{G_1A}$ + **AE** = $\mathbf{G_1E}$.

In the similar triangles, let $\angle EAB = ... = \theta$, and $\frac{AE}{AB} = k$, then the spiral similarity (k, θ) takes *AB* to *AE* and *CD* to *CG*. Thus AD_1 goes to AG_1. It follows easily that this similarity takes $\mathbf{D_1B}$, i.e. $\mathbf{D_1A}$ + **AB**, to $\mathbf{G_1A}$ + **AE**, that is, $\mathbf{G_1E}$.

If we translate *DA*, *HA* by vector **AC** we obtain D_2C, H_2C, and a similar argument shows that the spiral similarity $(k, -\theta)$ takes $\mathbf{D_2B}$ to $\mathbf{H_2F}$.

Now triangle BD_2D_1 is isosceles: BD is $\perp$ to D_2D_1, and D is the midpoint of that line, so BD_2, BD_1 are equal in length and are equally inclined to D_2D_1, i.e. to *CA*, although with opposite angles of rotation, say φ.

Consequently, the opposite spiral similarities give images G_1E, H_2F that are equal in length, and equally inclined to *CA*, at an angle $\theta + \varphi$, and the result follows.

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"*Statistics can be used to support anything - especially statisticians*" - Franklin Jones, **Women's Realm**