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Feedback: More on Hexagons with Opposite Sides Parallel

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Chris Bradley's article [1] on hexagons with opposite sides parallel reminded me of a simple related result I discovered a few months ago, which is not mentioned in his article. However, in all likelihood it is not new, but may perhaps have some interest for readers. Thanks to John Silvester from the Dept. of Mathematics, King's College, London for the following projective proof.

Theorem

If the midpoints of opposite sides of a hexagon, with opposite sides parallel, are connected, then the three lines are concurrent (see Figure 1).

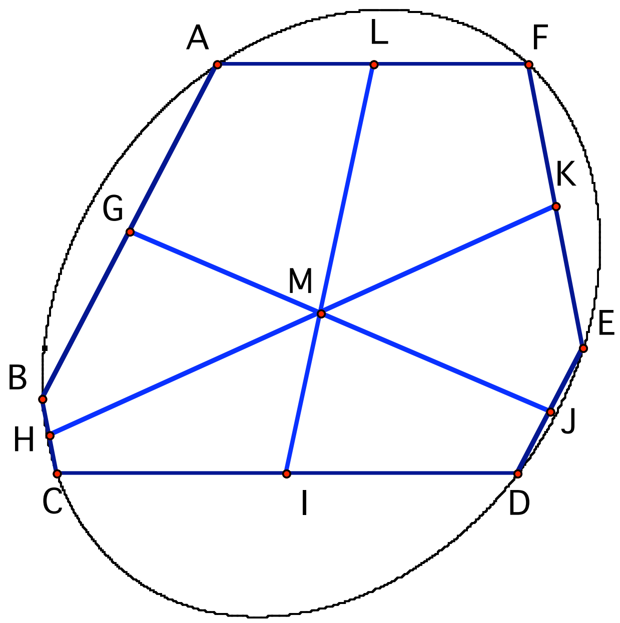

FIGURE 1

Proof

Let *ABCDEF* be a hexagon (in the projective plane), with opposite sides *AB*, *DE*; *BC*, *EF*; *CD*, *FA* meeting at *P*, *Q*, *R* respectively. If *P*, *Q*, *R* are collinear then (as we know) by the converse of Pascal's theorem, *ABCDEF* lie on a proper conic *S*. Let *G*, *H*, *I*, *J*, *K*, *L* be the harmonic conjugates of *P*, *Q*, *R*, *P*, *Q*, *R* with regard to *A*, *B*; *B*, *C*; *C*, *D*; *D*, *E*; *E*, *F*; *F*, *A* respectively. Then *GJ*, *HK*, *IL* are the polars of *P*, *Q*, *R* respectively, with regard to *S*. It follows that they are concurrent, at the pole *M* of the line *PQR*. If we now move to the Euclidean interpretation where the opposite sides of *ABCDEF* are parallel, then *PQR* is the line at infinity, so its pole *M* (if finite) is the centre of *S*; and the harmonic conjugates *G*, *H*, *I*, *J*, *K*, *L* are the midpoints of the various sides of the hexagon. But if *S* is a parabola, then

M is at infinity - this is the case where PQR touches S, at M - so in the Euclidean interpretation, the lines GJ, HK, IL are then parallel, rather than concurrent. (They are parallel to the axis of S.)

In the other case, where S is a parallel line-pair, it is not necessary to use projective geometry, and it is an easy exercise for the reader to show that the join of the mid-points of two parallel chords is either concurrent with the two lines of the line-pair (if they meet), or else is parallel to them (if they do not meet).

Also note that since the lines LI, GJ and HK are concurrent (in the projective plane), it follows from Brianchon's theorem that $GHIJKL$ has an inscribed conic.

Special cases

Of interest also is when the hexagon with opposite sides parallel degenerates into a pentagon with two pairs of parallel sides as shown in Figure 2 by letting vertices A and B coincide. Or by letting say vertices A and F, and C and D of the original hexagon $ABCDEF$ coincide as shown in Figure 3, we obtain a familiar result for parallelograms, namely, that the intersection of the diagonals and the lines connecting midpoints of opposite sides are concurrent.

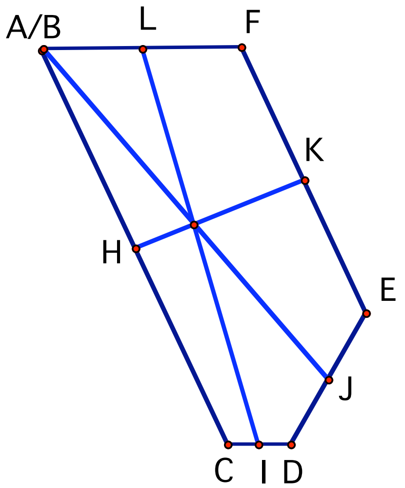

FIGURE 2

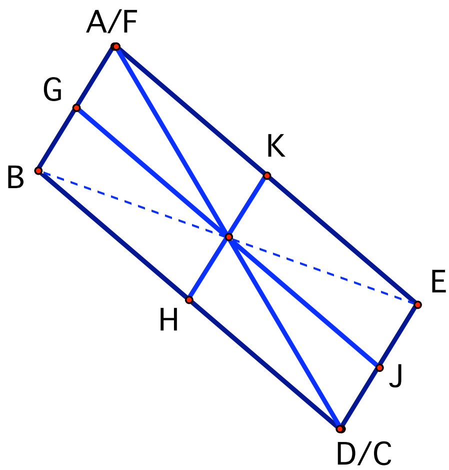

FIGURE 3

Finally, if we let say the vertices A and F, B and C, and D and E of the original hexagon $ABCDEF$ coincide, we obtain the familiar result of the concurrency of the medians of a triangle!

Note

Dynamic Geometry (*Sketchpad 4*) sketches in zipped format (Winzip) of the results discussed here can be downloaded directly from:

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http://mysite.mweb.co.za/residents/profmd/parahex.zip

(If not in possession of a copy of *Sketchpad 4*, these sketches can be viewed and manipulated with a free demo version of *Sketchpad 4* that can be downloaded from: http://www.keypress.com/sketchpad/sketchdemo.html)

Footnote

On 10 March 2007, I was kindly informed by Don Chakerian retired from the Math Dept at the University of California at Davis that the above result is posed as Exercise 7 on p. 281 of Veblen and Young, *Projective Geometry*, Vol. 2, Blaisdell Publishing Co., 1946.

References

1. C.J. Bradley, Hexagons with opposite sides parallel, *Math. Gaz.*, **90** (March 2006) pp. 57-67.

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