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The nine-point conic: a rediscovery and proof by computer

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A heuristic description is given of the rediscovery with *Sketchpad* of a less-well-known, but beautiful, generalization of the nine-point circle to a nine-point conic, as well as an associated generalization of the Euler line. The author's initial analytic geometry proofs, which made use of the symbolic algebra facility of the TI-92 calculator, are also presented.

1. Introduction

Although Euler was apparently the first person (in 1765) to show that the midpoints of the sides of a triangle and the feet of the altitudes determine a unique circle, it was not until 1820 that Brianchon and Poncelet showed that the three midpoints of the segments from the orthocentre to the vertices also lie on the same circle, hence its name, the nine-point circle [1]. The nine-point circle is often also referred to as the Euler or Feuerbach circle. (In 1822 Karl Feuerbach proved that the nine-point circle is tangent to the incircles and excircles of the triangle.)

A result closely associated with the nine-point circle is that of the Euler line, namely that the orthocentre (H), centroid (G), circumcentre (O) and nine-point centre (N) are collinear. Moreover, $HG = 2GO$ and $HN = 3NG$.

In October 2002, the author was investigating different ways of generalizing the nine-point circle with the aid of the dynamic geometry software *Sketchpad*. This resulted in discovering an interesting generalization to a nine-point ellipse (when all the points are on the sides of the triangle), or to a nine-point hyperbola (when some of the points lie on the extensions of the sides). After speaking to some colleagues who seemed to have no knowledge of it, and consulting some of the books the author had access to, he excitedly thought for a while he had made a novel discovery.

So imagine his disappointment when he contacted John Rigby (retired from Cardiff University), who promptly not only supplied him with a proof of his own, but pointed out that the result is mentioned and proved in [2] using projective methods. In fact, it turns out the result was merely a special case of a more general result known as the eleven-point conic (when the points lie on the extensions of the sides)! As someone who is very reliant on visualization, it was quite amazing

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(and humbling) to learn that this result was already discovered and proved in the 1890s [3], at a time when mathematicians did not yet have access to the wonderful kind of dynamic geometry software we have today.

During 2003, the author also managed to obtain a copy of John Silvester's excellent book, and much to his surprise the nine-point ellipse is also mentioned and elegantly proved there using affine geometry [4]. Here it is also explained how a six-point parabola can be obtained as a limiting case.

In this article, however, a description will be given of the experimental rediscovery of this beautiful result, which does not appear to be that well-known (largely due to the decline of geometry). Secondly, it will be shown how the author initially proved it using analytic (coordinate) geometry and utilizing the symbolic algebra ability of the TI-92 calculator. The nine-point ellipse result will also be linked to a generalization of the Euler line, which does not appear in the two references mentioned, but in all likelihood is not new.

2. Experimental discovery

The investigation started off by considering what happens if instead of the concurrent altitudes, one took any three concurrent cevians (lines from the vertices to the opposite sides). Constructing the midpoints of the segments from the cevian point H to the vertices as shown in figure 1, led to wondering whether there was any significance in them. Dynamically dragging and manipulating the triangle with *Sketchpad* for a while, it suddenly visually seemed to suggest that the feet of the cevians D, E and F, and the midpoints J, K and L all lie on an ellipse. This was immediately confirmed when a *Sketchpad* tool was used to draw an ellipse through any five of these points, the ellipse passing through the remaining sixth point. Delightfully surprising, it became apparent with further dragging that this ellipse always passed through the midpoints of the sides of triangle ABC. In other words, nine points in total lie on this uniquely determined ellipse!

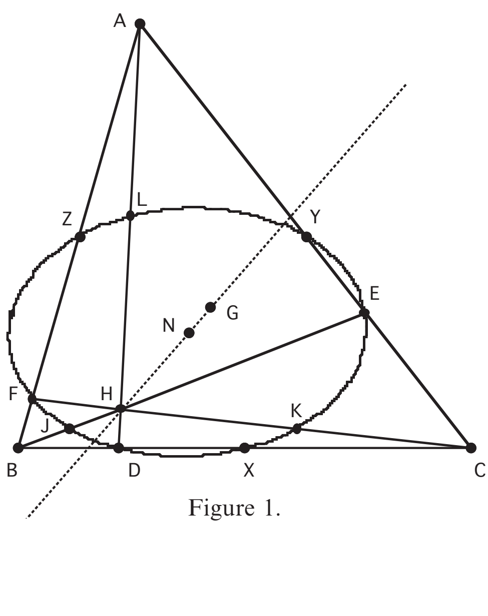

Figure 1.

Quite excited at this point, the centre N of the ellipse and the centroid G of triangle ABC was constructed, and it was found that not only were H, N and G collinear, but $HN = 3NG$. In other words, the result was also a generalization of the Euler line!

3. Proving affine results

Proving the result (for the points on the sides) was fairly routine as it is clearly an affine theorem, and one therefore only needed to choose an appropriate special case with which to show that it is true in general. For example, under affine transformations, angles, lengths and shapes are not preserved, but the following are some of the properties of a plane figure that remain invariant (compare [5] and [6]):

- the concurrency and collinearity of corresponding lines and points;
- the ratio in which a corresponding point divides a corresponding segment.

Moreover as shown in [7] and [8], the individual plane conics are all *affine equivalent*, which implies that using only affine transformations, any ellipse can be mapped onto any other ellipse, any hyperbola onto any other hyperbola, and any parabola onto any other parabola. (In fact, for the parabola, the similarities suffice – compare [9].)

4. Computer-aided proof

Therefore, without loss of generality, one could simply consider the right triangle ABC in figure 2 with angle $BAC = 30°$, angle $BCA = 60°$, and points B and C respectively placed at (0;0) and (1;0). Next

$$D\left(\frac{\sqrt{3}}{2+\sqrt{3}}; 0\right)$$

and

$$F\left(0; \frac{1}{\sqrt{3}}\right)$$

were arbitrarily chosen as the feet of the cevians AD and CF. Label H as the intersection of AD and CF, and draw the cevian BH to intersect AB at E, and label the midpoints J, K and L from H to the vertices as shown. Also label the midpoints of the sides of the triangle X, Y and Z as shown.

From trigonometry and coordinate geometry, the coordinates of points A, Z, X and Y were easily determined as follows:

$$A\left(0; \sqrt{3}\right), Z\left(0; \frac{\sqrt{3}}{2}\right), X\left(\frac{1}{2}; 0\right), Y\left(\frac{1}{2}; \frac{\sqrt{3}}{2}\right).$$

Next the coordinates of H was found by solving the simultaneous linear equations

$$y = -(2+\sqrt{3})x + \sqrt{3}$$

(line AD) and

$$y = -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$$

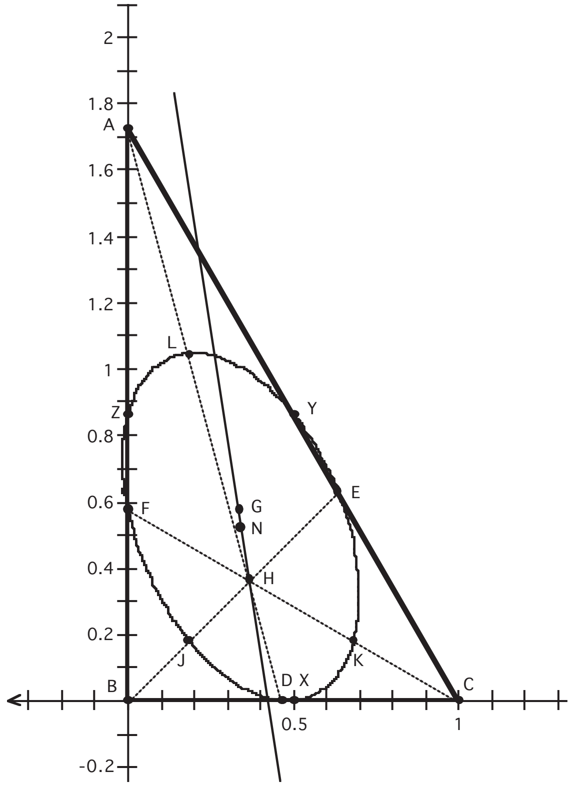

Figure 2.

(line *CF*) giving

$$H\left(\frac{1}{\sqrt{3}+1};\frac{1}{\sqrt{3}+1}\right).$$

By using Ceva's theorem, or otherwise from the intersection of lines *BH* and *AC*, the coordinates of *E* can be found as

$$E\left(\frac{\sqrt{3}}{1+\sqrt{3}};\frac{\sqrt{3}}{1+\sqrt{3}}\right).$$

The coordinates of the respective midpoints *J*, *K* and *L* of the segments *HB*, *HC* and *HA* are therefore

$$J\left(\frac{1}{2(\sqrt{3}+1)};\frac{1}{2(\sqrt{3}+1)}\right),\quad K\left(\frac{2+\sqrt{3}}{2(\sqrt{3}+1)};\frac{1}{2(\sqrt{3}+1)}\right)$$
$$\text{and}\quad L\left(\frac{1}{2(\sqrt{3}+1)};\frac{4+\sqrt{3}}{2(\sqrt{3}+1)}\right).$$

Now rewrite the equation of a general conic in the equivalent form below:

$$px^2+2qxy+ry^2+2sx+2ty+u=0$$
$$\Leftrightarrow 2q'xy+r'y^2+2s'x+2t'y+u'=-x^2$$

where $q'=(q/p)$, $r'=(r/p)$, etc.

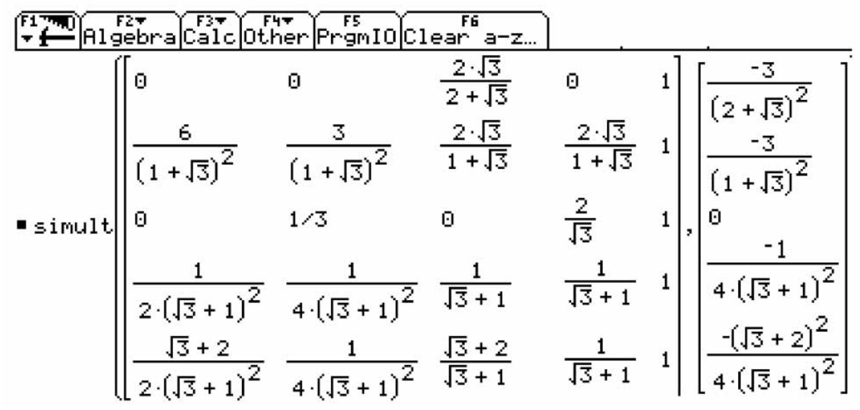

Figure 3.

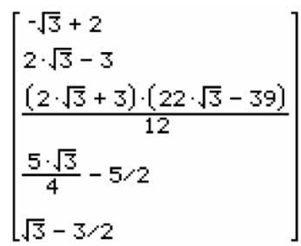

Figure 4.

Since five points uniquely determine a conic (as shown in [10] by using the invariant cross-ratio or by Pascal's theorem as in [11]), the equation of the conic could now be determined by the substitution of the coordinates of any five of the six points, say D, E, F, J, and K, and solving the system of five simultaneous linear equations. For the latter purpose, the symbolic algebra processing facility of the TI-92 calculator was ideally suited. By setting up a matrix as shown in figure 3, the solution was quickly obtained as shown in figure 4. (Note that figures 3 and 4 were compiled from two different TI-92 displays as everything does not fit into one display window, though one can observe the rest on the TI-92 simply by scrolling). The desired equation of the conic is therefore

$$x^2 + 2\left(-\sqrt{3}+2\right)xy + \left(2\sqrt{3}-3\right)y^2 + 2\left(\frac{(\sqrt{3}+3)(11\sqrt{3}-39)}{3}\right)x + \left(\frac{5\sqrt{3}}{2}-5\right)y + \left(\sqrt{3}-\frac{3}{2}\right) = 0.$$

Since $q^2 - pr = (-\sqrt{3}+2)^2 - (2\sqrt{3}-3) = -6\sqrt{3}+10 = -0.3923 < 0$, the conic is an ellipse. Using the TI-92, or any other symbolic algebra package, it is easy to verify by substitution into the above equation that the points L, X, Y, and Z also lie on this ellipse.

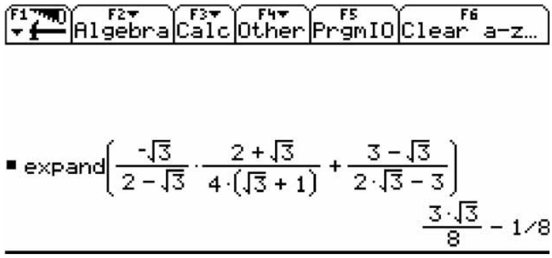

Figure 5.

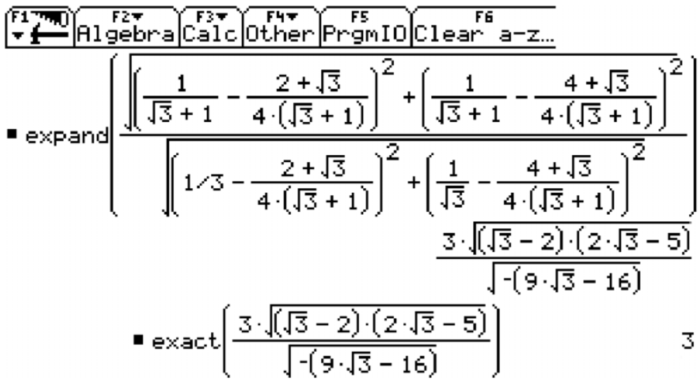

Figure 6.

By using the formula for the axes of symmetry and centre of a general ellipse from [5], or otherwise by symmetry arguments, the coordinates of the centre N of the ellipse are:

$$\left(\frac{2+\sqrt{3}}{4(\sqrt{3}+1)}; \frac{4+\sqrt{3}}{4(\sqrt{3}+1)}\right).$$

Solve the simultaneous equations given by the medians AX and CZ, or use other means, to find the coordinates of the centroid G of the triangle ABC as $(1/3; 1/\sqrt{3})$.

The equation of the line HG is given by

$$y = \frac{-\sqrt{3}}{2-\sqrt{3}}x + \frac{3-\sqrt{3}}{2\sqrt{3}-3}.$$

and the substitution of the x-coordinate of N into this equation using the TI-92 gives the simplified form as shown in figure 5. By rationalizing $4(\sqrt{3}+1)$, the denominator of the y-coordinate of N, one obtains the same simplified form, $3(\sqrt{3}-1)/8$, which proves that N lies on the line HG, and therefore that H, N and G are collinear.

Lastly, the ratio $HN/NG = 3$ can also easily be proved with the aid of the TI-92, and the application of the Pythagorean distance formula as shown in figure 6.

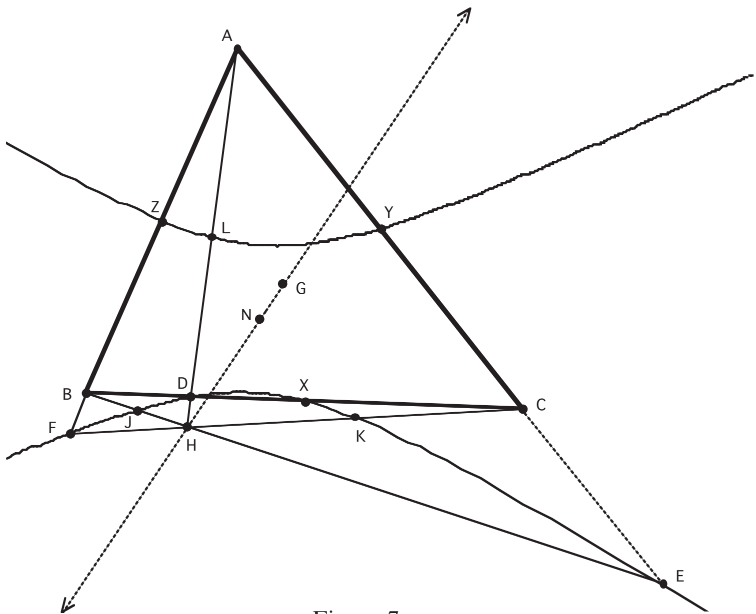

Figure 7.

5. Further extension

An interesting degenerate case is obtained when D, E and F coincide with the respective midpoints X, Y and Z of the sides, so that an *inscribed* six-point ellipse is obtained, but then the points H, N and G coincide so that the generalised Euler line is undefined. (The degenerate case is trivially true by symmetry if one considers the special case of an equilateral triangle — in which case the ellipse simply becomes the incircle.)

Furthermore, if we allow the concurrent cevians to fall outside the triangle, on the extensions of the sides, the ellipse can transform into a hyperbola as shown in figure 7. A six-point parabola can also be obtained as shown in [4] by taking the limit as H goes off to infinity. Therefore the result is more generally projective, and the further 11-point conic generalization follows directly from the projective equivalence of the conics (any conic can be mapped onto any other conic using suitable projectivities), and the invariance of collinearity under projections (also see [6] and [7]).

6. Concluding comments

This article has shown the value of computing technology not only in the visual rediscovery/discovery of the results and the provision of confidence to look for proofs, but also in handling the algebraic computation of analytic proofs with ease. In the absence of a more elegant approach, analytic techniques via computer algebra certainly provide a very efficient means of verification. The proofs presented here also only require a basic knowledge of analytic geometry, and no advanced knowledge of synthetic projective geometry.

However, on the negative side such analytic proofs usually provide little understanding of why the results are true. Although they provide confirmation, they often leave one with a sense of dissatisfaction: an unresolved need for deeper understanding. In essence, they often simply boil down to a brute force approach of manipulating algebraic equations and churning out equivalent equations and expressions until one achieves the desired result.

Acknowledgements

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Note

A Dynamic Geometry (*Sketchpad 4*) sketch in zipped format (Winzip) of the results discussed here can be downloaded directly from:
http://mzone.mweb.co.za/residents/profmd/9point.zip
(If not in possession of a copy of *Sketchpad 4*, this sketch can be viewed with a free demo version of *Sketchpad 4* that can be downloaded from:
http://www.keypress.com/sketchpad/sketchdemo.html)

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