Published in *Mathematics in School*, 31(5), Nov. 2002, p. 22.

A further Pythagorean variation on a Fibonacci theme

by Michael de Villiers

I was intrigued by the article by Pagni (2001) as it again illustrated the wonderful, almost limitless richness of the Fibonacci sequence which can provide students with many opportunities for original discoveries and creative work. In this article, Pagni firstly showed how taking any four consecutive Fibonacci numbers, say $F_n, F_{n+1}, F_{n+2}, F_{n+3}$, and letting $a = F_n F_{n+3}$, $b = 2F_{n+1}F_{n+2}$, then the number $c = \sqrt{(F_n F_{n+3})^2 + (2F_{n+1}F_{n+2})^2}$ is always an integer. The three integers *a*, *b* and *c* therefore form a Pythagorean Triple so that $c^2 = a^2 + b^2$.

By letting *x*, *y*, *x* + *y*, *x* + 2*y* represent any four consecutive Fibonacci numbers Pagni shows that $a^2 + b^2$ is always a perfect square, namely, $(x^2 + 2xy + 2y^2)^2$, and therefore that $c = x^2 + 2xy + 2y^2$. The proof is also valid for a generalised Fibonacci sequence where F_1 and F_2 are not necessarily chosen equal to 1. However, by rewriting *c* as $x^2 + 2xy + y^2 + y^2 = (x + y)^2 + y^2$, one can immediately see the following interesting relationship $(F_n F_{n+3})^2 + (2F_{n+1}F_{n+2})^2 = F_{n+1}^2 + F_{n+2}^2$ not pointed out by Pagni.

Perhaps more interesting though is that from the algebra, one can also see that this latter relationship easily generalises to a general Fibonacci type sequence with the construction rule $F_n = F_{n-k} + F_{n-2k}$ for $k \geq 1$ and $n > 2k$, and where $T_1, T_2, \ldots T_{2k}$ are chosen arbitrarily. For such a sequence, the following always holds. Take any four numbers $F_n, F_{n+k}, F_{n+2k}, F_{n+3k}$, and let $a = F_n F_{n+3k}$, $b = 2F_{n+k}F_{n+2k}$, then the number $c = \sqrt{(F_n F_{n+3k})^2 + (2F_{n+k}F_{n+2k})^2}$ is always an integer. The three integers *a*, *b* and *c* therefore form a Pythagorean Triple so that $c^2 = a^2 + b^2$. In addition, we also have the following interesting relationship $(F_n F_{n+3k})^2 + (2F_{n+k}F_{n+2k})^2 = F_{n+k}^2 + F_{n+2k}^2$.

For example, let *k* = 2 and arbitrarily choose the first four terms as 3, 1, 7 and 4. We then obtain the following sequence:

3, 1, 7, 4, 10, 5, 17, 9, 27, 14, 44, 23, 71, 37, 115, 60, 186, 97, 301, ...

Now choose $F_2 = 1$, $F_4 = 4$, $F_6 = 5$, $F_8 = 9$. Then $c = \sqrt{(1 \times 9)^2 + (2 \times 4 \times 5)^2} = 41$.
But $F_4^2 + F_6^2 = 4^2 + 5^2 = 16 + 25 = 41$.

Note, however, that unlike with the regular Fibonacci sequence, the number 41 is not a member of the same sequence, and therefore the elegant Fibonacci relationship $(F_n F_{n+3})^2 + (2F_{n+1}F_{n+2})^2 = (F_{2n+3})^2$ pointed out by Pagni does not apply in this general case.

Reference

Pagni, D. (2001). Fibonacci meets Pythagoras. *Mathematics in School*, 30(4), 39-40.

Keywords: Fibonacci, Pythagoras.

Author

Michael de Villiers, Mathematics Education, University Durban-Westville, South Africa.

E-mail: profmd@mweb.co.za

Homepage: http://mzone.mweb.co.za/residents/profmd/homepage.html