A simple proof of an interesting Fibonacci generalization

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It is reasonably well known that the ratios of consecutive terms of a Fibonacci series converge to the golden ratio. This note presents a simple, complete proof of an interesting generalization of this result to a whole family of ‘precious metal ratios’.

1. Introduction

A beautiful property of the Fibonacci series is that the ratios of consecutive terms converge to the golden ratio. By generalizing the recursive formula $T_n + T_{n+1} = T_{n+2}$ for a Fibonacci series to the general formula $T_n + T_{n+k} = T_{n+k+1}$, where $k=1,2,...$, De Villiers [1] made the interesting discovery that for each member of this family of series, the ratios of consecutive terms converge to the positive roots of $x^{k+1} - x^k - 1 = 0$. However, based on the assumption that $\lim_{n \to \infty} \left( T_{n+k+1}/T_{n+k} \right)$ exists, only a partial proof to this result was given.

De Villiers [2] suggested a simple proof for the case where $k$ is odd, with the suggestion that it could be generalized to also cover the case where $k$ is even. What follows is a generalization of this approach, and provides a complete proof of the result.

2. Some preliminaries

From the equation $x^{k+1} = x^k + 1$ (1) we can deduce that

$$x^k(x - 1) = 1 \Rightarrow x^k = \frac{1}{x - 1}$$

and therefore, to solve equation (1) is to solve the system

$$\left\{ \begin{array}{l} f(x) = \frac{1}{x - 1} \\
g(x) = x^k \end{array} \right.$$  

(2)

that is to say, to find the intersection of the curves defined in (2) for $k=1,2,...$

If $k$ is even, the graph of $g(x) = x^k$ is a curve of parabolic type, and the intersection of the two curves is given in figure 1.

Then, system (2) has only one real solution $x = M > 1$, which approaches 1 as the value of $k$ increases.

If $k$ is odd, the graph of $g(x) = x^k$ is a curve of cubic type, and the intersection of the two curves is given in figure 2.

Note that system (2) admits only two real roots $M$ and $\lambda$ such that $M > 1$ and $-1 < \lambda < 0$. These solutions tend to 1 and $-1$ respectively when $k$ increases.
The other roots are simple complex numbers and the modulus lie between $|\lambda|$ and $M$ if $k$ is an odd number and between $M/2$ and $M$ if $k$ is even.

In short: the roots of equation (1) are one positive real number $M$ and $k$ complex solutions $\lambda_j$ of the form $\lambda_j = a_j + ib_j$ in such form that if $k$ is odd, one of these roots lacks an imaginary part and its real part is negative. These $k$ complex roots can be expressed in exponential form as $\lambda_j = r_je^{i\phi_j}$, where

$$r_j = \sqrt{a_j^2 + b_j^2} \quad \text{and} \quad \tan\phi_j = \frac{b_j}{a_j}$$

and such that $M > r_j$ for $j = 1, 2, \ldots, k$.

**Theorem.** If $T_n$ is the $n$th term of a sequence with the property $T_n + T_{n+k} = T_{n+k+1}$, then for $k \geq 0$

$$\lim_{n \to \infty} \frac{T_{n+k+1}}{T_{n+k}} = M$$

where $M$ is the positive root of $x^{k+1} - x^k - 1 = 0$. 
Proof. The preceding sequence is equivalent to \( T_n = T_{n-1} + T_{n-k-1} \) (it is enough to substitute \( n + k + 1 \) by \( n \)) and this one is a difference equation which characteristic equation is \( x^n = x^{n-1} + x^{n-k-1} \) or, that is the same, \( x^{k+1} = x^k + 1 \). Taking into account the discussion in the preceding section, we will have that the solution of the difference equation is (see [3])

\[
T_n = a_1M^n + \sum_{j=2}^{k+1} a_j \lambda_j^n
\]

(3)

On the other hand,

\[
\lim_{n \to \infty} \frac{T_{n+k+1}}{T_{n+k}} = \lim_{n \to \infty} \frac{T_n}{T_{n-1}}
\]

so, taking into account formula (3) it is

\[
\lim_{n \to \infty} \frac{T_n}{T_{n-1}} = \lim_{n \to \infty} \frac{a_1M^n + \sum_{j=2}^{k+1} a_j \lambda_j^n}{a_1M^{n-1} + \sum_{j=2}^{k+1} a_j \lambda_j^{n-1}} = \lim_{n \to \infty} \frac{a_1 + \sum_{j=2}^{k+1} a_j (\lambda_j/M)^n}{a_1(1/M) + \sum_{j=2}^{k+1} a_j (\lambda_j/M)^{n-1}(1/M)}
\]

\[
= \lim_{n \to \infty} \frac{a_1 + \sum_{j=2}^{k+1} a_j (r_j/M)^n e^{i\theta_j}}{(a_1/M) + \sum_{j=2}^{k+1} a_j (r_j/M)^{n-1}(e^{i\theta_j})/r_j} = \frac{a_1}{a_1/M} = M
\]

because, since \( M > r_j \),

\[
\lim_{n \to \infty} \left( \frac{r_j}{M} \right)^n = 0.
\]

References


The inscribed sphere of an \( n \)-simplex

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The centre and radius of the inscribed \( n \)-dimensional sphere of an \( n \)-simplex are derived using elementary linear algebra.

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