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A Fibonacci Generalization and its Dual

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"*Symmetry as wide or as narrow as you may define it, is one idea by which man through the ages has tried to comprehend, and create order, beauty and perfection.*"

- Hermann Weyl

INTRODUCTION

Duality is a special kind of symmetry. In everyday language, a common duality exists between antonyms such as hot and cold, tall and short, love and hatred, male and female, etc. Basically, the one concept is defined by and understood in terms of the other, and together they form a whole which complement and enrich each other.

In mathematics there are often important dualities between certain concepts and operators. For example, in projective geometry we find an interesting duality between the following concepts:

vertices (points)	-	sides (lines)
inscribed in a conic	-	circumscribed around a conic
collinear	-	concurrent

Two theorems or configurations are called *dual* if the one may be obtained from the other by replacing each concept and operator by its dual concept or operator. Some other mathematical topics where duality occurs are Boolean algebra, tessellations, polyhedra, trigonometry, etc. If a general duality exists, then *all* the theorems of that particular topic occur in pairs, each similar to the other and identical in structure, except for the *interchange* of dual concepts. In such a case therefore the *dual* of any true theorem, is another true theorem. In fact, it is unnecessary to prove the dual results since their proofs can be obtained by simply writing down the proofs of the original results word by word, replacing only relevant concepts with their corresponding duals. The establishment of a general duality is therefore, apart from its aesthetic appeal, also very economical from a logical point of view.

In this article an interesting duality between *addition* and *multiplication* of terms to produce sequences or series will be discussed, and examples will be presented that could provide a valuable source for investigative or enrichment work for students at the high school or under-graduate level. At the most basic level this duality is apparent from the fact that both operations are commutative, as well as associative. In other words, for any mathematical expression based only on these properties, the two operations are interchangeable; ie. dual.

DUALITY BETWEEN ARITHMETIC SERIES & GEOMETRIC PRODUCT

Yeshurun (1978) has pointed out a useful duality between arithmetic and geometric sequences which is apparently not very well known. Consider for example:

(1) an arithmetic sequence: $a; a+d; a+2d; \ldots; a+(n-1)d$

(2) a geometric sequence: $a; ar; ar^2; \ldots; ar^{n-1}$

By comparing these two examples it should be clear that they are essentially constructed in the same way. For the arithmetic sequence, a constant number is *added* to the first term to produce the second term, then to the second to produce the third term, etc. For the geometric sequence, however, a constant number is *multiplied* with the first term to produce the second term, then to the second to produce the third term, etc. So clearly the one sequence can be obtained from the other by simply interchanging the *addition* of a constant number with the *multiplication* by a constant number, and are therefore *dual*. A further comparison of the n -th term of each sequence also shows that this interchange results in a corresponding interchange between a *linear* function and an *exponential* function of n. For example, in the case of the arithmetic sequence the constant number d is *multiplied by a factor* (n - 1) whereas in the geometric sequence the constant number r is *raised to the power* (n - 1).

This duality extends to arithmetic *series* and geometric *products* as follows:

(1) an arithmetic series: $S = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d)$

(2) a geometric product: $P = a \times (ar) \times (ar^2) \times \ldots \times (ar^{n-1})$

To derive a formula for an arithmetic series, we usually write two versions below each other as follows:

$$S = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d)$$
$$S = (a+(n-1)d) + \ldots + (a+2d) + (a+d) + a$$

Then by adding these together, simplifying, and calling the n -th term q, one easily arrives at the following formula for an arithmetic series: $S = \frac{n}{2}(a+q)$. Similarly, one can derive a formula for a geometric product by writing two versions below each other:

$$P = a \times (ar) \times (ar^2) \times \ldots \times (ar^{n-1})$$
$$P = (ar^{n-1}) \times \ldots \times (ar^2) \times (ar) \times a$$

Then by multiplying these together, simplifying, and calling the n -th term q, one easily arrives at the following formula for a geometric product: $P = (aq)^{\frac{n}{2}}$.

Here the duality between the two formulae is again clearly apparent. In the case of the arithmetic series the *sum* of the first and last term is *multiplied by a factor* $\frac{n}{2}$ whereas in the geometric product the *product* of the first and last term is *raised to the power* $\frac{n}{2}$.

A FIBONACCI GENERALIZATION

The well-known Fibonacci series, namely:

$$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 + \ldots$$

can easily be constructed by the rule $T_n + T_{n+1} = T_{n+2}$, where the n-th term is called T_n. Of course, one does not have to start with $T_1 = 1$ and $T_2 = 1$, but any arbitrarily chosen T_1 and T_2 would do. If we call the sum to n terms S_n, then the Fibonacci series has the following interesting property: $T_2 + S_n = T_{n+2}$.

Conversely, if we construct a series according to the rule $C + S_n = T_{n+2}$, then it will have the property $T_n + T_{n+1} = T_{n+2}$ for all $n > 1$. For example, arbitrarily choose $T_1 = 1$, $T_2 = 2$ and $C = 3$, then according to the rule $C + S_n = T_{n+2}$:

$C + S_1 = 3 + 1 = 4 = T_3$

$C + S_2 = 3 + 3 = 6 = T_4$

$C + S_3 = 3 + 7 = 10 = T_5$; etc.

This gives the series: $1 + 2 + 4 + 6 + 10 + 16 + 26 + 42 + \ldots$ which clearly has the property $T_n + T_{n+1} = T_{n+2}$ for all $n > 1$. If however C is chosen equal to T_2 then it is also true for $n = 1$.

The Fibonacci series can further be considered as a special case of a whole family of series which can be constructed by simple variations in the above construction rules. For example one could let one's students investigate the following sets of rules:

<u>Term addition rule</u>	<u>Sum addition rule</u>
$T_n + T_n = T_{n+1}$	$C + S_n = T_{n+1}$
$T_n + T_{n+1} = T_{n+2}$	$C + S_n = T_{n+2}$
$T_n + T_{n+2} = T_{n+3}$	$C + S_n = T_{n+3}$
$T_n + T_{n+3} = T_{n+4}$	$C + S_n = T_{n+4}$

Before reading any further the reader is encouraged to first construct a few examples according to the above rules. A heuristic description of the Lakatosian way in which a similar investigation by a high school teacher and his class lead to the following two generalizations in relation to this family of series is given in De Villiers (in press):

Theorem 1

If T_n is the nth term and S_n is the *sum* to n terms of these terms, then for all $n > 1$:

$C + S_n = T_{n+k+1} \Rightarrow T_n + T_{n+k} = T_{n+k+1}$ where C is any real number and $k \geq 0$.

Proof

The proof is based on the assumptions (a) and (b) below (which follow automatically from the notation):

(a) $S_n = S_{n-1} + T_n$ ... $n > 1$

(b) $T_{n+k+1} = C + S_n \Leftrightarrow T_{n+k} = C + S_{n-1}$ (n is replaced by n - 1).

$T_{n+k+1} = C + S_n$

$\Rightarrow$ $T_{n+k+1} = C + S_{n-1} + T_n$ ... from (a)

$\Rightarrow$ $T_{n+k+1} = T_{n+k} + T_n$ ... from (b)

Note that S_0 becomes implicitly defined as $S_0 = T_{k+1} - C$ (from assumption (b)) in the construction of such series. However from assumption (a) we have that $S_0 = S_1 - T_1 = 0$. Assumption (a) will therefore be true for $n = 1$ only if we choose T_{k+1} in such a way that S_0 becomes 0 in $S_0 = T_{k+1} - C$; therefore T_{k+1} must be chosen equal to C. In other words, if we choose $T_{k+1} = C$, assumption (a) would be valid for $n = 1$ and therefore the conclusion $T_{n+k+1} = T_{n+k} + T_n$ would then be true for all n.

Theorem 2

If T_n is the nth term and S_n is the *sum* to n terms of these terms, then for all n:

$T_n + T_{n+k} = T_{n+k+1} \Rightarrow T_{k+1} + S_n = T_{n+k+1}$ where $k \geq 0$.

Proof

Firstly write the consecutive terms of the series as the following differences:

$T_1 = T_{k+2} - T_{k+1}$

$T_2 = T_{k+3} - T_{k+2}$

$T_3 = T_{k+4} - T_{k+3}$

.

.

.

$T_{n-1} = T_{k+n} - T_{k+n-1}$

$T_n = T_{k+n+1} - T_{k+n}$

Then adding up the left and right columns respectively, we find the desired result $S_n = T_{k+n+1} - T_{k+1}$ or $S_n + T_{k+1} = T_{n+k+1}$.

A DUAL FIBONACCI GENERALIZATION

Using the duality between arithmetic series and geometric products mentioned earlier, one can now immediately formulate the following two dual theorems to Theorems 1 & 2. (Although it is not necessary to give the proofs, they will be given below simply to illustrate the duality).

Theorem 3

If T_n is the nth term and P_n is the *product* to n terms of these terms, then for all $n > 1$: $C \times P_n = T_{n+k+1} \Rightarrow T_n \times T_{n+k} = T_{n+k+1}$ (where $k \geq 0$).

Proof

The proof is based on the assumptions (a) and (b) below (which follow automatically from the notation):

(a) $P_n = P_{n-1} \times T_n$ ... $n > 1$

(b) $T_{n+k+1} = C \times P_n \Leftrightarrow T_{n+k} = C \times P_{n-1}$ (n is replaced by n - 1).

$T_{n+k+1} = C \times P_n$

$\Rightarrow$ $T_{n+k+1} = C \times P_{n-1} \times T_n$ ... from (a)

$\Rightarrow$ $T_{n+k+1} = T_{n+k} \times T_n$... from (b)

Note that P_0 becomes implicitly defined as $P_0 = T_{k+1} \div C$ (from assumption (b)) in the construction of such series. However from assumption (a) we have that $P_0 = P_1 \div T_1 = 1$. Assumption (a) will therefore be true for $n = 1$ only if we choose T_{k+1} in such a way that P_0 becomes 1 in $P_0 = T_{k+1} \div C$; therefore T_{k+1} must be chosen equal to C. In other words, if we choose $T_{k+1} = C$, assumption (a) would be valid for $n = 1$ and therefore the conclusion $T_{n+k+1} = T_{n+k} \times T_n$ would then be true for all n.

Theorem 4

If T_n is the nth term and P_n is the *product* to n terms of these terms, then for all n:

$T_n \times T_{n+k} = T_{n+k+1} \Rightarrow T_{k+1} \times P_n = T_{n+k+1}$ (where $k \geq 0$).

Proof

Firstly write the consecutive terms of the product as the following quotients:

$$T_1 = \frac{T_{k+2}}{T_{k+1}}$$

$$T_2 = \frac{T_{k+3}}{T_{k+2}}$$

$$T_3 = \frac{T_{k+4}}{T_{k+3}}$$

.

.

.

$$T_{n-1} = \frac{T_{k+n}}{T_{k+n-1}}$$

$$T_n = \frac{T_{k+n+1}}{T_{k+n}}$$

Then multiplying up the left and right columns respectively, we find the desired result $P_n = T_{k+n+1} \div T_{k+1}$ or $P_n \times T_{k+1} = T_{n+k+1}$.

Examples

Let us consider an example of Theorem 3 for $k = 1$ (which is the dual to the Fibonacci series itself). Arbitrarily choose $T_1 = 2$, $T_2 = 3$ and $C = 2$. Then $T_3 = C \times P_1 = 2 \times 2 = 4$, $T_4 = C \times P_2$

= $2\times 6 = 12$, etc., giving us the series: $2\times 3\times 4\times 12\times 48\times 576\times 27648\times 15925248$ … Here we clearly have as before $T_1\times T_2 \neq T_3$, but $T_2\times T_3 = T_4$, $T_3\times T_4 = T_5$, $T_4\times T_5 = T_6$, etc.

Let us also consider an example of Theorem 4 for $k = 1$ (which is also the dual to the Fibonacci series itself). Arbitrarily choose $T_1 = 2$, $T_2 = 3$. Then $T_3 = T_1\times T_2 = 6$, $T_4 = T_2\times T_3 = 18$, etc., giving us the series: $2\times 3\times 6\times 18\times 108\times 1944\times 209952\times 408146688\times$ … Here we clearly have as before $P_1\times T_2 = T_3$, $P_2\times T_2 = T_4$, $P_3\times T_2 = T_5$, etc.

THE GOLDEN & OTHER RATIOS

In a golden rectangle, the rectangle obtained by removing a square from one end is similar to the original rectangle (see Figure 1). The ratio of the length to the width of such a rectangle is called the *golden ratio* and is often denoted by the symbol ϕ. This ratio $\phi = a/b$ is defined by:

$$\frac{a}{b} = \frac{b}{a-b}.$$

Cross multiplying and then dividing by b^2 gives:

$$\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right) - 1 = 0.$$

So the golden ratio is the positive root of the quadratic equation:

$$x^2 - x - 1 = 0$$

and has a value of 1.61803 (correct to 5 decimals).

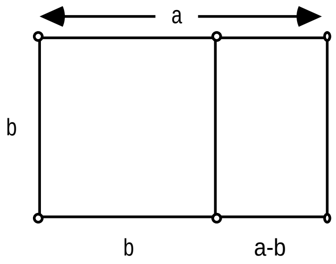

Figure 1

A truly surprising result is the relationship of the Fibonacci sequence with the golden ratio. For example, the limit of the quotients of adjacent terms of the Fibonacci sequence is the golden ratio, ie.:

$$\lim_{n\to\infty} \frac{T_{n+1}}{T_n} = \phi.$$

Since convergence is fast, it is a good activity to let students compute these ratios using a calculator or a computer and watching them approach ϕ. What about the ratios of adjacent terms for the family of series we have discussed earlier? Do they also approach a limit? Are there corresponding limits for the dual Fibonacci products?

Let us consider a case where $k = 2$ with the property $T_n + T_{n+2} = T_{n+3}$:

1 + 1 + 1 + 2 + 3 + 4 + 6 + 9 + 13 + 19 + 28 + 41 + 60 + 88 + 129 + 189 + 277 + ...

Here we have the following ratios (correct to four decimals):

$$\frac{T_{11}}{T_{10}} = \frac{28}{19} = 1.4736; \frac{T_{12}}{T_{11}} = 1.4642; \frac{T_{13}}{T_{12}} = 1.4634; \frac{T_{14}}{T_{13}} = 1.4666; \text{ etc.}$$

From the repetition of the first two decimals, we clearly already have convergence correct to two decimal places. It is left to the reader to explore this and other cases further.

In the preceding case we were looking for a number d so that $T_n \times d = T_{n+1}$. In the dual case, we therefore need to look for a number so that $(T_n)^d = T_{n+1}$. In other words, for the dual case we need to consider the ratios: $\frac{\log T_{n+1}}{\log T_n}$. Let us now consider the example of a dual Fibonacci product discussed in the previous paragraph, namely: $2 \times 3 \times 6 \times 18 \times 108 \times 1944 \times 209952 \times 408146688 \times \ldots$

Here we have the following ratios (correct to four decimals):

$$\frac{\log T_5}{\log T_4} = 1.6199; \frac{\log T_6}{\log T_5} = 1.6173; \frac{\log T_7}{\log T_6} = 1.6183; \frac{\log T_8}{\log T_7} = 1.6179; \text{ etc.}$$

From the repetition of the first two decimals, we clearly already have convergence to the golden ratio correct to two decimal places. It is a good exercise to let one's students explore this and other cases further, and to allow them to discover the rather surprising generalization below. Technology like graphics calculators with table facilities, or a spreadsheet on computer, could be very useful in this respect. In what follows a partial proof of these observations will be given that should be accessible to high school students.[1]

Theorem 5

If T_n is the nth term of a sequence with the property: $T_n + T_{n+k} = T_{n+k+1}$, then for $k \geq 0$: $\lim_{n\to\infty} \frac{T_{n+k+1}}{T_{n+k}} = \alpha$ where α is the positive root of $x^{k+1} - x^k - 1 = 0$.

Proof

If we assume that $\lim_{n\to\infty} \frac{T_{n+k+1}}{T_{n+k}} = \alpha$ exists, then we have the following:

[1] A complete proof can be found in an article by Sergio Falcon (2002) in *IJMEST*, and which can be downloaded directly from http://mysite.mweb.co.za/residents/profmd/fibonacci.pdf

$$T_{n+k+1} = T_{n+k} + T_n$$
$$\frac{T_{n+k+1}}{T_{n+k}} = 1 + \frac{T_n}{T_{n+k}}$$
$$\frac{T_{n+k+1}}{T_{n+k}} = 1 + \frac{T_n}{T_{n+1}} \bullet \frac{T_{n+1}}{T_{n+2}} \bullet \ \ldots \ \bullet \frac{T_{n+k-1}}{T_{n+k}}$$
$$\lim_{n\to\infty}\left(\frac{T_{n+k+1}}{T_{n+k}}\right) = 1 + \lim_{n\to\infty}\left(\frac{T_n}{T_{n+1}} \bullet \frac{T_{n+1}}{T_{n+2}} \bullet \ \ldots \ \bullet \frac{T_{n+k-1}}{T_{n+k}}\right)$$
$$\alpha = 1 + \frac{1}{\alpha^k}$$
$$\alpha^{k+1} - \alpha^k - 1 = 0$$

From the above it is therefore clear that if $\lim_{n\to\infty} \frac{T_{n+k+1}}{T_{n+k}} = \alpha$ exists, α is a root of the polynomial $x^{k+1} - x^k - 1 = 0$. To prove the existence of this limit in general is however a matter that goes beyond the scope of this article. For *k* is odd, the equation $x = 1 + \frac{1}{x^k}$ has two real solutions, and it is easy to generalize the approach used by Schielack (1987). However, for *k* is even (where there is only one real solution), and the more general case which includes the consideration of complex roots, it appears that one would have to utilize an approach similar to that of approach of Niven, Zuckerman & Montgomery (1991: 493-499).

Furthermore, students who explored it empirically may have noticed that these ratios α_k start at 2 for *k* = 0, and then appear to decrease towards a limiting value of 1 as *k* increases. This observation can also easily be explained as follows. For *k* = 0, the series has the rule $T_n + T_n = T_{n+1}$, obviously giving us the constant ratio $\frac{T_{n+1}}{T_n} = 2$, which of course corresponds to the solution of the equation $x = 1 + \frac{1}{x^k}$ for this value of *k*. By letting *k* increase in the latter equation, it now follows that $\frac{1}{x^k}$ decreases and therefore the root α must correspondingly decrease. Finally, taking the limit as $k \to \infty$ of the same equation, we obtain $\alpha = 1$.

It is also interesting to ask: what geometric interpretation can be given to these ratios α_k (which incidentally, I fondly refer to as the "*precious metal ratios*")? Clearly if we start with a rectangle with sides *a* and *b* where $a \geq b$, then $\left(\frac{a}{b}\right)^{k+1} - \left(\frac{a}{b}\right)^k - 1 = 0$. Multiplying through by b^{k+1} and rearranging we obtain: $\left(\frac{a}{b}\right)^k = \frac{b}{a-b}$. Geometrically, this therefore means that after the square with sides *b* is removed, the rectangle obtained must be similar to a rectangle with sides a^k and b^k. Examples of corresponding rectangles for *k* = 0, *k* = 1 and *k* = 2 are respectively shown in

Figures 2a, 2b and 2c. It is also obvious that as k increases b approaches a and the rectangle tends towards a square.

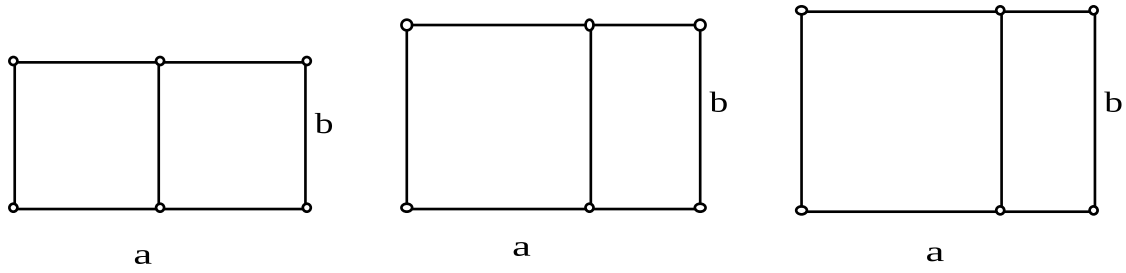

Figure 2

Let us now consider the dual of Theorem 5 and its proof.

Theorem 6

If T_n is the nth term of a sequence with the property: $T_n \times T_{n+k} = T_{n+k+1}$, then for $k \geq 0$: $\lim_{n\to\infty} \frac{\log T_{n+k+1}}{\log T_{n+k}} = \alpha$ where α is the positive root of $x^{k+1} - x^k - 1 = 0$.

Proof

Consider the property $T_n \times T_{n+k} = T_{n+k+1}$. By taking logarithms on both sides, it can be transformed into $\log T_n + \log T_{n+k} = \log T_{n+k+1}$. This equation is in form equivalent to the recursive formula $T_n + T_{n+k} = T_{n+k+1}$ discussed in the previous paragraph, and it can therefore be shown in the same way that if the limit $\lim_{n\to\infty} \frac{\log T_{n+k+1}}{\log T_{n+k}} = \alpha$ exists, it is a solution of the given polynomial.

ANOTHER GENERALIZATION AND ITS DUAL

Coleman (1989) has shown another interesting generalization of the Fibonacci sequence by using the general rule $T_n + B\times T_{n+1} = T_{n+2}$ where B is a positive integer, and showing that the ratios $\frac{T_{n+1}}{T_n}$ approach the positive root of $x^2 - Bx - 1 = 0$ as n becomes large.

In the same way we can construct a dual with the rule $T_n \times (T_{n+1})^B = T_{n+2}$ (where B is a positive integer), and easily show that the ratios $\frac{\log T_{n+1}}{\log T_n}$ approach the positive root of $x^2 - Bx - 1 = 0$ as n becomes large.

Recently, Siddiqui (1995), a high school student, gave the following result that for series generated by the following rule: $T_n + T_{n+1} + T_{n+2} = T_{n+3}$ we have:

$(T_3 - T_1) + 2S_n = T_{n+3} - T_{n+1}$.

It is left to the reader to verify that the corresponding dual holds for products created by the rule: $T_n \times T_{n+1} \times T_{n+2} = T_{n+3}$, namely:

$$\left(\frac{T_3}{T_1}\right) \times P_n^{\ 2} = \frac{T_{n+3}}{T_{n+1}}.$$

To not disappoint those who know that I can seldom resist ending a lecture (or an article) without posing a final question or two for further investigation, I would therefore in conclusion like to pose to the reader the further investigation of the aforementioned theorems in relation to sequences and series with the general property $A \times T_n + B \times T_{n+k} = T_{n+k+1}$, and their duals. Another challenging investigation is to explore the aforementioned theorems in relation to sequences and series with the general property $A \times T_n + B \times T_{n+1} = T_{n+k}$ ($k > 1$), and their duals. This is of course not too mention exploring the further generalization of the wealth of other properties of the standard Fibonacci series!

> "*Mathematics is the only infinite human activity. It is conceivable that humanity could eventually learn everything in physics or biology. But humanity certainly won't ever be able to find out everything in mathematics, because the subject is infinite.*"
>
> \- Paul Erdös

> "*It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never-satisfied man is so strange - if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is conquered, stretches out his arms for another.*"
>
> \- Karl Gauss

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