An interesting generalization for the Fermat-Torricelli point of a triangle involves changing the equilateral triangles on the sides as follows (a proof is given in [1]):

**Theorem 1**

If triangles $DBA$, $ECB$ and $FAC$ are constructed (outwardly or inwardly) on the sides of any triangle $ABC$ so that $\angle DAB = \angle CAF$, $\angle DBA = \angle CBE$ and $\angle ECB = \angle FCA$ then lines $DC$, $EA$ and $FB$ are concurrent.

A corollary to Theorem 1 which shows that in a sense it is really only a generalization of Ceva's theorem, is that
\[
\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = 1.
\]

Recently the following different generalization of the Fermat-Torricelli point was proved (amongst other properties) in [2] by using complex numbers:

**Theorem 2**

If $L_1$ and $L_2$, $M_1$ and $M_2$, and $N_1$ and $N_2$ are pairs of points respectively on the sides $BC$, $CA$ and $AB$ of any triangle $ABC$ such that
\[
\frac{BL_1}{LC} = \frac{CL_2}{LA} = \frac{CM_1}{MA} = \frac{AM_2}{MC} = \frac{AN_1}{NA} = \frac{BN_2}{NB} = \frac{N_2A}{N_1B},
\]
and equilateral triangles are $DN_1N_2$, $XN_2L_1$, $EL_2L_1$, $YL_2M_1$, $FM_2M_1$ and $ZM_2N_1$ are constructed (outwardly or inwardly) on the sides of the hexagon $N_1N_2L_1L_2M_1M_2$, then lines $DY$, $EZ$ and $FX$ are concurrent.

(Notes: (1) In [2] the unnecessary restrictions are given that the ratios in which the pairs of points divide the sides of triangle $ABC$ must be smaller than 1, and that the equilateral triangles need to be constructed outwardly. (2) This result is also true if the pairs of points lie on the extensions of the sides of triangle $ABC$ provided that the three triangles with outer vertices $X$, $Y$ and $Z$ are oppositely
situated to the three triangles with outer vertices \( D, E \) and \( F \); eg. if the former are inwardly then the latter must be outwardly, or vice versa).

Interestingly, the above two results can be combined to provide the following further generalization:

**Theorem 3**

If \( L_1 \) and \( L_2 \), \( M_1 \) and \( M_2 \), and \( N_1 \) and \( N_2 \) are pairs of points respectively on the sides \( BC, CA \) and \( AB \) of any triangle \( ABC \) such that

\[
\frac{BL_1}{CL_1} = \frac{CL_2}{BL_2} = \frac{AM_1}{M_1C} = \frac{AN_1}{N_1B} = \frac{BN_1}{N_1A},
\]

and triangles \( DN_1N_2, \ XN_2L_1, \ EL_1L_2, \ YL_2M_1, \ FM_1M_2 \) and \( ZM_2N_1 \) are constructed (outwardly or inwardly) on the sides of the hexagon \( N_1N_2L_1L_2M_1M_2 \) so that \( \angle DN_1N_2 = \angle M_1M_2F = \angle XL_2N_2 = \angle YL_2M_1, \ \angle DN_2N_1 = \angle L_2L_1E = \angle YM_1L_2 = \angle ZM_2N_1 \) and \( \angle EL_1L_2 = \angle FM_1M_2 = \angle ZN_2M_2 = \angle XN_2L_1 \), then lines \( DY, EZ \) and \( FX \) are concurrent.

**Proof**

The proof is surprisingly simple. Consider Figure 1. From the given ratios

\[
\frac{AM_1}{M_1C} = \frac{AN_1}{N_1B} \quad \text{and} \quad \frac{BN_1}{N_1A} = \frac{CM_1}{M_1A},
\]

it respectively follows that \( N_1M_1 \parallel BC \) and \( N_1M_2 \parallel BC \), and therefore \( N_1M_2 \parallel N_2M_1 \parallel BC \). Similarly, \( N_2L_1 \parallel N_2L_2 \parallel AC \) and \( M_1L_2 \parallel M_1L_1 \parallel AB \). The following pairs of opposite triangles are therefore homothetic (corresponding sides parallel and similar): \( \Delta DN_1N_2 \) and \( \Delta YL_2M_1 \), \( \Delta EL_1L_2 \) and \( \Delta ZM_2N_1 \), and \( \Delta FM_1M_2 \) and \( \Delta XN_2L_1 \). Thus the lines connecting the corresponding vertices of these pairs of homothetic triangles are respectively concurrent at the points \( R, P \) and \( Q \). From the parallelness of corresponding sides it now follows that triangles \( PQR \) and \( M_2QM_1 \) are similar and that the similarity with center \( Q \) which maps triangle \( M_2QM_1 \) to triangle \( PQR \) also maps point \( F \) to point \( F' \) (on line \( QF \)). Therefore triangles \( FPR \) and \( FM_1M_i \) are similar. In the same way triangles \( DFP \) and \( EQR \) , respectively similar to triangles \( DN_1N_2 \) and \( EL_1L_2 \) (and with \( D' \) and \( E' \) respectively on lines \( RD \) and \( PE \), can be constructed. From Theorem 1, it now follows that lines \( DR, EP \) and \( FQ \) are concurrent, and thus also lines \( DY, EZ \) and \( FX \).

From the corollary in Theorem 1 and the similarity of the pairs of opposite triangles, the following corollary also holds:

\[
\frac{ND_1}{YN_1} \times \frac{NL_1}{EL_1} \times \frac{LE_1}{YM_1} \times \frac{YM_1}{FM_2} \times \frac{FM_2}{ZN_2} \times \frac{ZN_2}{ZM_2} = 1.
\]

**References**


Figure 1