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A further generalization of the Fermat-Torricelli point

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An interesting generalization for the Fermat-Torricelli point of a triangle involves changing the equilateral triangles on the sides as follows (a proof is given in [1]):

Theorem 1

If triangles *DBA*, *ECB* and *FAC* are constructed (outwardly or inwardly) on the sides of any triangle *ABC* so that $\angle DAB = \angle CAF$, $\angle DBA = \angle CBE$ and $\angle ECB = \angle FCA$ then lines *DC*, *EA* and *FB* are concurrent.

A corollary to Theorem 1 which shows that in a sense it is really only a generalization of Ceva's theorem, is that

$$\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = 1.$$

Recently the following different generalization of the Fermat-Torricelli point was proved (amongst other properties) in [2] by using complex numbers:

Theorem 2

If L_1 and L_2, M_1 and M_2, and N_1 and N_2 are pairs of points respectively on the sides *BC*, *CA* and *AB* of any triangle *ABC* such that

$$\frac{BL_1}{L_1C} = \frac{CL_2}{L_2B} = \frac{CM_1}{M_1A} = \frac{AM_2}{M_2C} = \frac{AN_1}{N_1B} = \frac{BN_2}{N_2A},$$

and equilateral triangles are DN_1N_2, XN_2L_1, EL_1L_2, YL_2M_1, FM_1M_2 and ZM_2N_1 are constructed (outwardly or inwardly) on the sides of the hexagon $N_1N_2L_1L_2M_1M_2$, then lines *DY*, *EZ* and *FX* are concurrent.

(Notes: (1) In [2] the unnecessary restrictions are given that the ratios in which the pairs of points divide the sides of triangle *ABC* must be smaller than 1, and that the equilateral triangles need to be constructed outwardly. (2) This result is also true if the pairs of points lie on the extensions of the sides of triangle *ABC* provided that the three triangles with outer vertices *X*, *Y* and *Z* are oppositely situated to the three triangles with outer vertices *D*, *E* and *F*; eg. if the former are inwardly then the latter must be outwardly, or vice versa).

Interestingly, the above two results can be combined to provide the following further generalization:

Theorem 3

If L_1 and L_2, M_1 and M_2, and N_1 and N_2 are pairs of points respectively on the sides *BC*, *CA* and *AB* of any triangle *ABC* such that

$$\frac{BL_1}{L_1C}=\frac{CL_2}{L_2B}=\frac{CM_1}{M_1A}=\frac{AM_2}{M_2C}=\frac{AN_1}{N_1B}=\frac{BN_2}{N_2A},$$

and triangles DN_1N_2, XN_2L_1, EL_1L_2, YL_2M_1, FM_1M_2 and ZM_2N_1 are constructed (outwardly or inwardly) on the sides of the hexagon $N_1N_2L_1L_2M_1M_2$ so that $\angle DN_1N_2=\angle M_1M_2F=\angle XL_1N_2=\angle YL_2M_1$, $\angle DN_2N_1=\angle L_2L_1E=\angle YM_1L_2=\angle ZM_2N_1$ and $\angle EL_2L_1=\angle FM_1M_2=\angle ZN_1M_2=\angle XN_2L_1$, then lines *DY*, *EZ* and *FX* are concurrent.

Proof

The proof is surprisingly simple. Consider Figure 1. From the given ratios $\frac{AM_2}{M_2C}=\frac{AN_1}{N_1B}$ and $\frac{BN_2}{N_2A}=\frac{CM_1}{M_1A}$, it respectively follows that $N_1M_2//BC$ and $N_2M_1//BC$, and therefore $N_1M_2//N_2M_1//BC$. Similarly, $N_2L_1//N_1L_2//AC$ and $M_1L_2//M_2L_1//AB$. The following pairs of opposite triangles are therefore homothetic (corresponding sides parallel and similar): ΔDN_1N_2 and ΔYL_2M_1, ΔEL_1L_2 and ΔZM_2N_1, and ΔFM_1M_2 and ΔXN_2L_1. Thus the lines connecting the corresponding vertices of these pairs of homothetic triangles are respectively concurrent at the points *R*, *P* and *Q*. From the parallelness of corresponding sides it now follows that triangles PQR and M_2QM_1 are similar and that the similarity with center *Q* which maps triangle M_2QM_1 to triangle PQR also maps point *F* to point *F'* (on line *QF*). Therefore triangles *F'PR* and FM_2M_1 are similar. In the same way triangles *D'PQ* and *E'QR* , respectively similar to triangles DN_1N_2 and EL_1L_2 (and with *D'* and *E'* respectively on lines *RD* and *PE*), can be constructed. From Theorem 1, it now follows that lines *D'R*, *E'P* and *F'Q* are concurrent, and thus also lines *DY*, *EZ* and *FX*.

From the corollary in Theorem 1 and the similarity of the pairs of opposite triangles, the following corollary also holds:

$$\frac{N_1D}{DN_2}\times\frac{N_2X}{XL_1}\times\frac{L_1E}{EL_2}\times\frac{L_2Y}{YM_1}\times\frac{M_1F}{FM_2}\times\frac{M_2Z}{ZN_1}=1.$$

References

[1] M. de Villiers, A generalization of the Fermat-Torricelli point. *The Mathematical Gazette*, Vol 79, No 485 (July 1995), 374-378.

[2] A generalization of Napoleon's theorem. (Problem 1493 proposed by J. Fukuta; published solution by O. Lossers). *Mathematics Magazine*, Vol 70, No 1 (Feb 1997), 70-73.

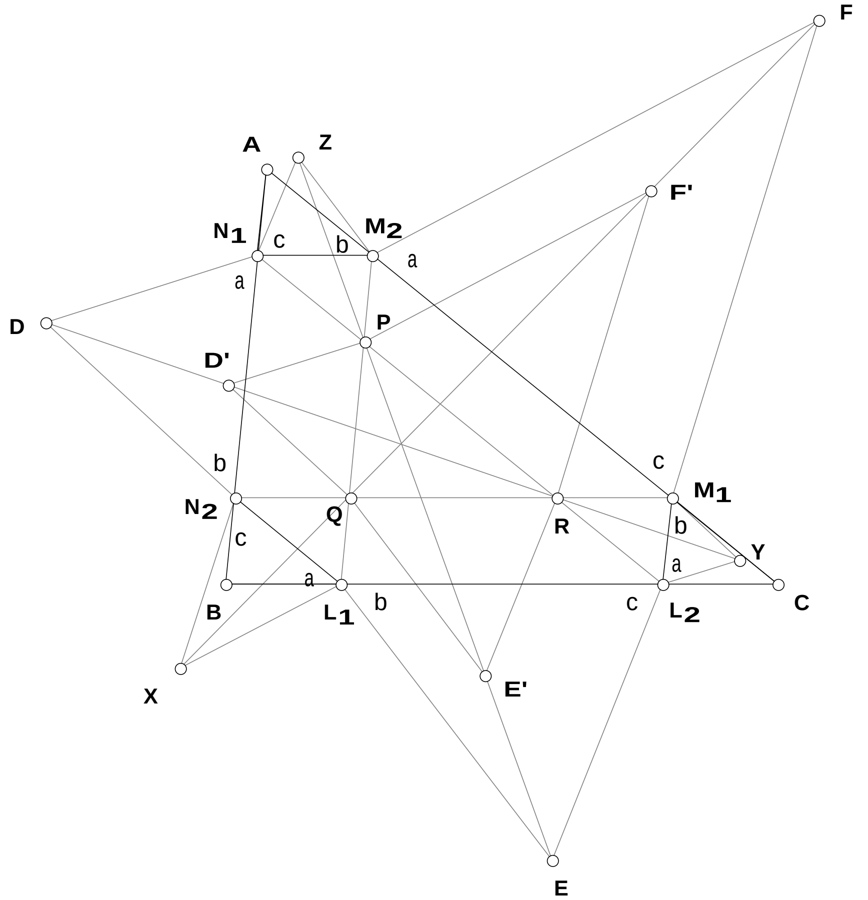

Figure 1