

A Generalization of the Nine-point circle and Euler line

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"Now I will have less distraction." - Leonhard Euler, referring to losing the sight of one eye, quoted in *Mathematical Circles* (Howard Eves, 1969)

Introduction

To most people, including some mathematics teachers, geometry is synonymous with ancient Greek geometry, especially as epitomized in Euclid's *Elements* of 300 BC. Sadly, many are not even aware of the significant extensions and investigations of Apollonius, Ptolemy, Pappus, and many others until about 320 AD. Even more people are completely unaware of the major developments that took place in synthetic Euclidean plane geometry from about 1750-1940, and more recently again from about 1990 onwards (stimulated in no small way by the current availability of dynamic geometry software).

The purpose of this article is therefore to give a brief historical background to the discovery of the Nine-point circle and the Euler line, and a simple, but possibly new generalization and proof of the latter, that may be of interest to teachers and students.

High School Background

The reader is reminded of the following three classic concurrency results from Euclid's *Elements* that are fortunately still mentioned (though seldom with proof) in a few South African high school textbooks. However, since these results are no longer required "theorems" for the final matric examination from about the late 1980's, it is likely that most teachers have simply ignored teaching them, thus producing a generation of children unacquainted with these remarkable results from our geometric heritage.

- (1) The medians (lines from the vertices to the midpoints of the opposite sides) of a triangle are concurrent at the *centroid* (centre of gravity) of a triangle (see Figure 1).
- (2) The altitudes (perpendicular lines from the vertices to the opposite sides) of a triangle are concurrent at the *orthocentre* (see Figure 2).
- (3) The perpendicular bisectors of the sides (lines through the midpoints of the sides and perpendicular to them) of a triangle are concurrent at the *circumcentre*, which is

the centre of the circle through the three vertices (see Figure 3).

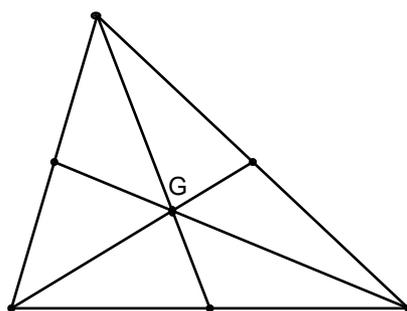


Figure 1: Centroid

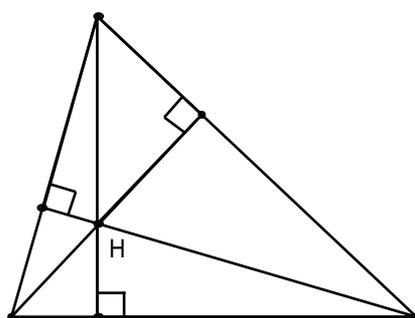


Figure 2: Orthocentre

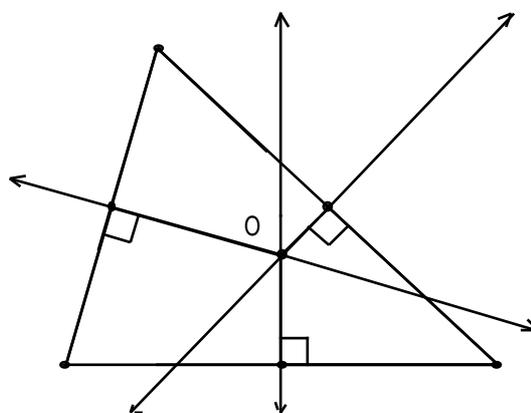


Figure 3: Circumcentre

Ceva's Theorem

In 1678 an Italian mathematician named Giovanni Ceva discovered a surprising generalization of the altitude and median (and angle bisector) concurrencies, namely, that if in any triangle, line segments AD , BF and CE are concurrent (with D , F and E respectively on sides BC , AC and AB), then $\frac{AF}{FC} \times \frac{CD}{DB} \times \frac{BE}{EA} = 1$. Conversely, if $\frac{AF}{FC} \times \frac{CD}{DB} \times \frac{BE}{EA} = 1$, then line segments AD , BF and CE are concurrent (see Figure 4). In Ceva's honour, the line

segments AD , BF and CE joining the vertices of a triangle to any given points on the opposite sides, are called *cevians*.

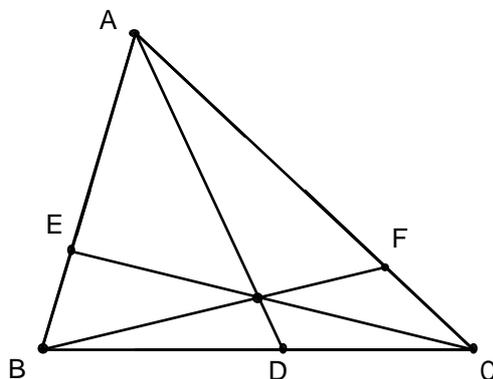


Figure 4: Ceva's Theorem

Ceva's theorem is a very important and useful theorem that has to form part of the standard armoury of any high school student aspiring to be competitive at the Third Round level of the SA Mathematics Olympiad. Students who participate in the workshops and Summer School of the Mathematical Talent Search organized under the auspices of the SA Mathematical Society (SAMS) are well acquainted with this result, as are all the SA team members of the International Mathematics Olympiad (IMO).

Homothetic Polygons

Another valuable result that is usually also well known to successful Mathematics Olympiad contestants is the following theorem: If two polygons are *homothetic* (that is similar and their corresponding sides are parallel), then the lines connecting corresponding vertices are concurrent at their centre of similarity (see Figure 5).

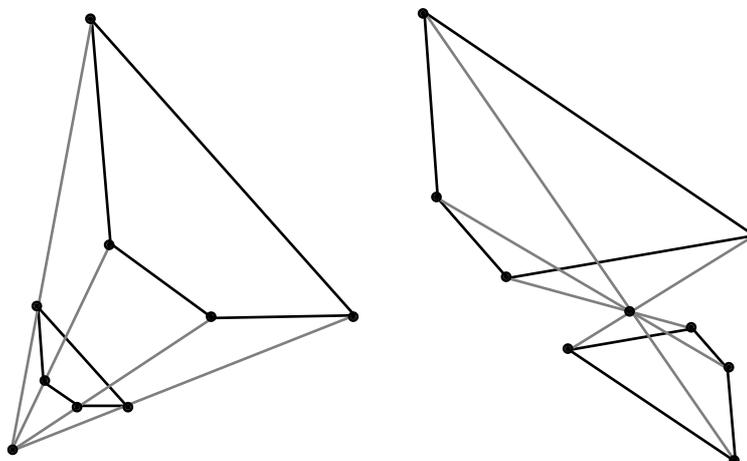


Figure 5: Homothetic Polygons

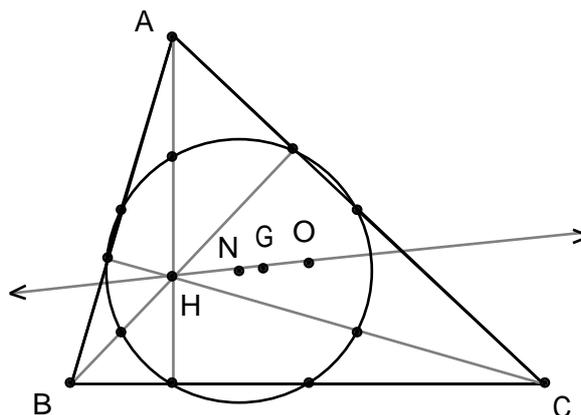


Figure 6: Nine-Point Circle & Euler Line

The Nine-point Circle and Euler Line

Although Leonhard Euler was apparently the first person in 1765 to show that the midpoints of the sides of a triangle and the feet of the altitudes determine a unique circle, it was not until 1820 that Brianchon and Poncelet showed that the three midpoints of the segments from the orthocentre to the vertices also lie on the same circle, hence its name, the *nine-point circle* (see Figure 6). The nine-point circle is often also referred to as the Euler circle in honour of Euler. It is also sometimes called the Feuerbach circle in honour of Karl Feuerbach who in 1822 proved the stunning theorem that the nine-point circle is tangent to the incircles and excircles of the triangle!

A result closely associated with the nine-point circle is that of the Euler line (which Euler presumably discovered more or less simultaneously), namely that the orthocentre (H), centroid (G), circumcentre (O) and the centre of nine-point circle (N) are collinear. Moreover, $HG = 2GO$ and $HN = 3NG$.

The historical background referred to above is widely available in standard historical resource books like Boyer (1968), Kramer (1970), etc. Books such as these ought to be regularly consulted by teachers and lecturers in order to bring a much-needed historical perspective to mathematics in the classroom. For classic synthetic proofs of the results mentioned above, readers can for example consult any advanced geometry textbooks like Coxeter & Greitzer (1967) or Posamentier (2002).

Experimental Discovery

In October 2002, I was wondering how one might generalize the nine-point circle and started off by considering what happens if instead of the concurrent altitudes, one took any three concurrent cevians (lines from the vertices to the opposite sides). Next I constructed

the midpoints of the segments from the cevian point H to the vertices as shown in Figure 7, wondering whether there was any significance in them. Dynamically dragging and manipulating the triangle with *Sketchpad* for a while, it suddenly visually seemed to suggest that the feet of the cevians D , E and F , and the midpoints J , K and L all lie on an ellipse. This was immediately confirmed when I used a *Sketchpad* tool to draw an ellipse through any five of these points, the ellipse passing through the remaining sixth point. Much to my surprise, and delight, I next noticed with further dragging that this ellipse always passed through the midpoints of the sides of triangle ABC (and turned into a hyperbola if D , E and F was dragged onto the extensions of the sides of the triangle). In other words, nine points in total lie on this uniquely determined conic!

After laboring through long analytic geometry proofs (with the aid of the symbolic computer algebra of the TI-92), I later found out, much to my dismay, that the discovery was not novel at all, and was already known in the 1890's appearing in some projective geometry texts (Russell, 1893:212). It also appears in standard projective geometry texts such as Baker (1922:41-42), which were required study material for entrance exams into the mathematical doctoral programs of Oxford and Cambridge in the 1920's and 1930's. It seems quite sad that such a beautiful projective geometry result has become forgotten and neglected.

This nine-point conic result, however, contains a generalization of the Euler line as a corollary, which does not appear in any of the three references mentioned, and an internet search has also provided no explicit mention of it in the mathematical literature. Furthermore, my initial analytic proofs of the nine-point conic and the Euler generalization (see De Villiers, in press), although confirming the results, do not provide any satisfactory insight into why the results are true. In contrast, the proof given further down for the Euler generalization is not only synthetic, but also explanatory in terms of similarity.

For the sake of completeness, the nine-point conic theorem and Euler line generalization are now formally stated, and a proof of the latter given.

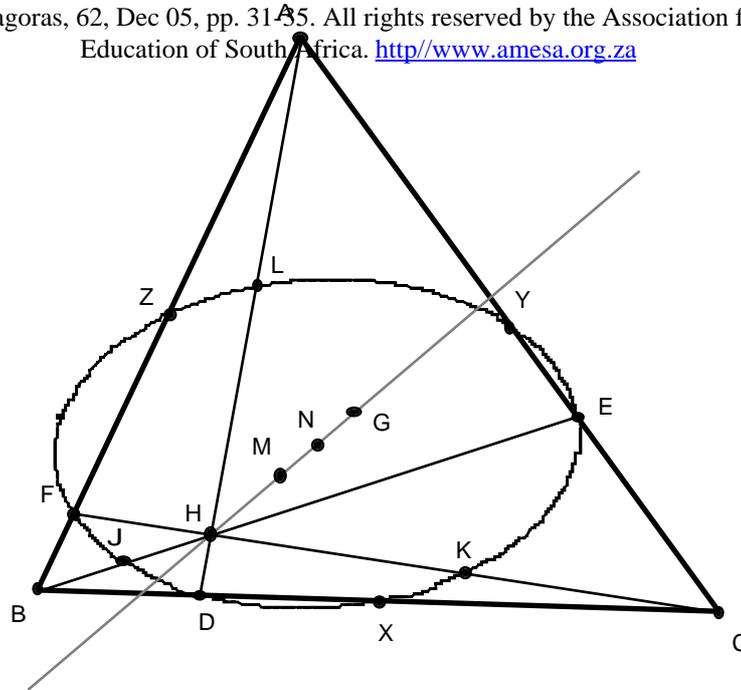


Figure 7: Nine-point Conic and Euler Line Generalization

The nine-point conic

Given any triangle ABC , and three cevians concurrent in H , then the feet of the cevians (D , E and F), the midpoints of the sides of the triangle (X , Y and Z), and the respective midpoints L , J and K of the segments HA , HB and HC , lie on a conic (Figure 7).

(Note that an ellipse is obtained when the feet of all the cevians are on the sides of the triangle, but when some of the feet of the cevians lie on the extensions of the sides, the conic becomes a hyperbola. Silvester (2001:214-215) also explains how a six-point parabola can (theoretically) be obtained as a limiting case as the cevian point H is dragged off towards infinity.)

Euler Line generalization

Given the above configuration for any triangle ABC , then the centre N of the conic, the centroid G of ABC and the point of cevian concurrency H , are collinear, and $HN = 3NG$. (Note that the circumcentre O of triangle ABC does not necessarily lie on this general Euler line. However, note that just as with the nine-point circle, the centre N of the nine-point conic, is the common midpoint of segments XL , YJ and ZK - see proof below).

Proof

Construct the point M as the image of G in N under a half-turn. Since G is the centroid of XYZ (and ABC), and N is the common centroid (midpoint and centre of gravity) of XL , YJ and ZK (as they are the respective diagonals of parallelograms $XXLZ$, $JXYL$ and $ZJKY$), it follows that M is the centroid of JKL . And since BCA is the image of the enlargement of JKL from H with scale factor 2, it follows that H, M, N and G are all collinear, and that $HG = 2HM$, $MN = NG$ which implies that $HN = 3NG$.

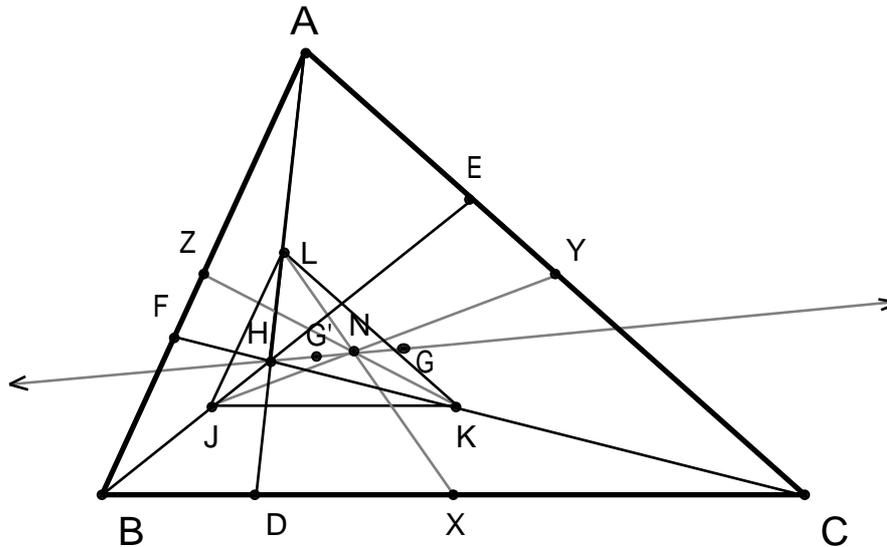


Figure 8: Further Euler Line generalization

Further Euler Line generalization

The above Euler line generalization generalizes even further as follows. Given any triangle ABC with midpoints of the sides X , Y and Z and three cevians concurrent in H as shown in Figure 8. With H as centre of similarity and scale factor $\frac{1}{k}$, construct triangle LJK similar to ABC . Let N be the centre of similarity between LJK and the median triangle XYZ . Then H, N and G are collinear, and $HN = \frac{3}{k-1} NG$.

Proof

Construct the centroid G' of triangle LJK . Since ABC maps onto LJK under the similarity situated at H , it follows that $\frac{HG}{HG'} = k$ and H, G' and G are collinear. Since the median triangle is also similar to ABC under a half-turn around G with a scale factor of $\frac{1}{2}$, it

follows that LJK is similar to XYZ with a scale factor $\frac{2}{k}$. More over, LJK is homothetic to XYZ . Therefore lines LX , JY and KZ are concurrent at N (with $2XN = kNL$, etc.). Then since the centroid of the median triangle coincides with the centroid G of ABC , and XYZ maps onto LJK under the half-turn and similarity situated at N , it follows that $\frac{NG}{NG'} = \frac{k}{2}$ and G', N and G are collinear. Since the straight line through G and G' is unique, it follows that H, G', N and G are collinear.

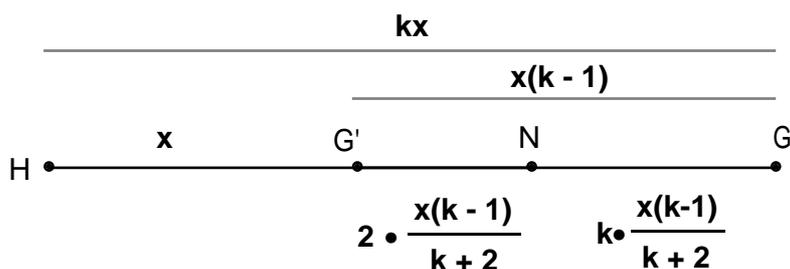


Figure 8: Ratios between segments

Let $HG' = x$, then from the ratios into which HG is divided as shown in Figure 8, it follows

$$\text{that } \frac{HN}{NG} = \frac{3xk}{k+2} \times \frac{k+2}{kx(k-1)} = \frac{3}{k-1}.$$

Looking Back

Instead of respectively using parallelograms and homothetic polygons to prove XL , YJ and ZK concurrent at N for the Euler line generalization and Further Euler line generalization, Ceva's theorem could be used. Further note that both Euler generalizations can be viewed as theorems about quadrilateral $ABCH$. An interesting physical interpretation of both generalizations is to consider finding the centroid N of point masses at A , B , C and H . For example, for the first Euler line generalization, consider unit masses at A , B , C and H . Then N is the centroid of 2 masses at X and L , etc., and also the centroid of 3 masses at G and 1 at H ; hence $HN = 3NG$. Similarly, for the Further Euler generalization, consider unit masses at A , B and C and mass $k - 1$ at H .

Since the Further Euler generalization no longer involves a conic, it is perhaps less interesting than the Euler generalization, which it generalizes. Indeed, this is often the case with generalization, since the general case frequently involves fewer properties than the special case.

Concluding Comment

As mathematicians we have an educational obligation to share new developments in our discipline with our students to combat the pervasive misconception that mathematics is a sterile and dead subject. Euclidean plane geometry is particularly suited as students can easily be led to some visual appreciation even without formal proof. More over, it may just stimulate their own creativity and inspire them to engage in some mathematical research themselves. The availability of dynamic geometry software also encourages a kind of experimental approach in which it is easy to make and check conjectures that lies well within the means of average students, and not just a select few.

Acknowledgements

1. This article is dedicated in memory of the recently deceased Martyn Cundy, co-author of the famous geometry classic *Mathematical Models* (published in 1951), who kindly suggested to me the elegant proof of the Euler line generalization, and which led to the Further Euler line generalization.
2. This research was partially funded by a National Research Foundation (NRF) grant from the Spatial Orientation & Spatial Insight (SOSI) Project, Pretoria, GUN No. 2050502.

Note: A Dynamic Geometry (*Sketchpad 4*) sketch in zipped format (Winzip) of the results discussed here can be downloaded directly from:

<http://mysite.mweb.co.za/residents/profmd/9pointeuler.zip>

(This sketch can also be viewed with a free demo version of *Sketchpad 4* that can be downloaded from: <http://www.keypress.com/sketchpad/sketchdemo.html>)

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