

## Recycling Cyclic Polygons Dynamically

Michael de Villiers

### Introduction

In the November 2005 issue, Chris Pritchard (2005) presented an interesting property of (convex) cyclic  $2n$ -gons, namely, that the two sums of alternate angles are equal to  $(n - 1)180^\circ$ . A natural mathematical question to ask is whether the converse is also true? (Compare De Villiers, 1996, pp. 65, 167 or De Villiers, 2003, pp.82-84, 181). For example, given any (convex)  $2n$ -gon with the two sums of alternate angles equal to  $(n - 1)180^\circ$ , is it necessarily cyclic?

Educationally, it is always a good idea to challenge one's students to first explore conjectures experimentally by means of dynamic geometry. For example, ask them to construct a  $2n$ -gon that meets the premise (i.e. the sum of alternate angles equal to  $(n - 1)180^\circ$ ), and then to check by dragging whether the conclusion (i.e. that it is cyclic) is always true. In the event that the conjecture is true, the value of such experimentation lies in providing the confidence to look for a proof, and might also sometimes suggest a way of proving it. Of course, if the conjecture is false, a counter-example will hopefully be produced.

### Experimental Exploration

Let's first consider a quadrilateral where the sum of one pair of opposite angles is supplementary (i.e.  $180^\circ$ ). How do we construct a dynamic quadrilateral with opposite (or alternate) angles always supplementary?

To achieve this, firstly construct a straight line as shown in Figure 1, dividing it into two supplementary angles  $DCA$  and  $DCB$  as shown. Then irrespectively of how  $D$  is dragged, these two angles will obviously remain supplementary.

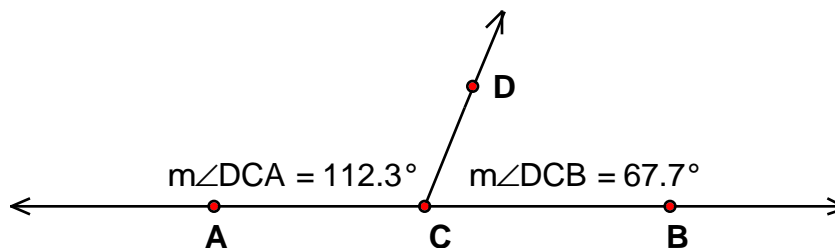


Figure 1

The next step is to construct two opposite angles  $HEF$  and  $HGF$  correspondingly equal to  $DCA$  and  $DCB$ . For example, as shown in Figure 2 using *Sketchpad*, construct a ray from  $E$ , then select angle  $DCA$  and choose **Mark Angle** from the *Transform* menu. Select  $E$  and choose **Mark Center** from the *Transform* menu. Then by selecting the ray from  $E$ , and choosing **Rotate** from the *Transform* menu, the ray will be rotated exactly by the marked angle  $DCA$ .

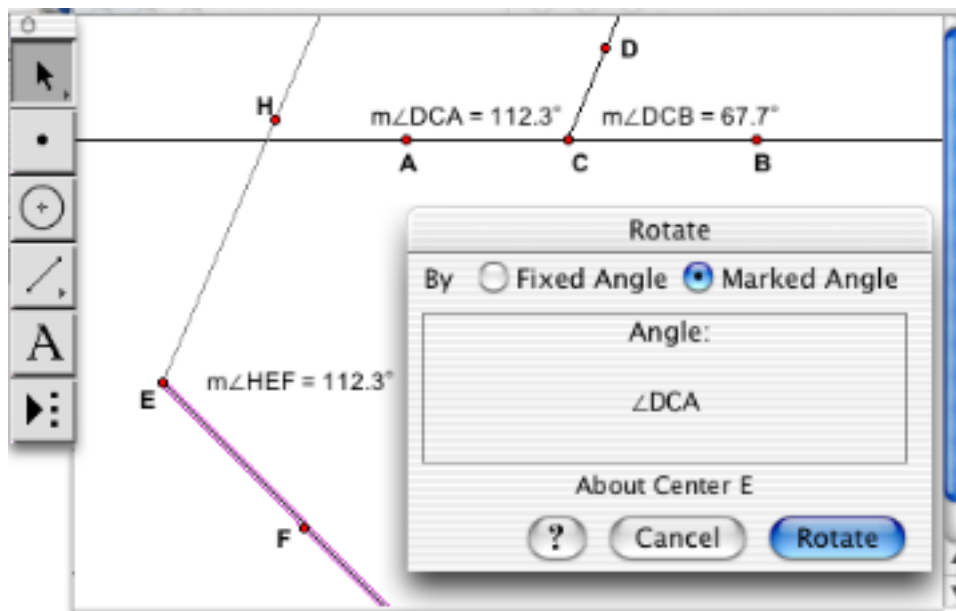


Figure 2

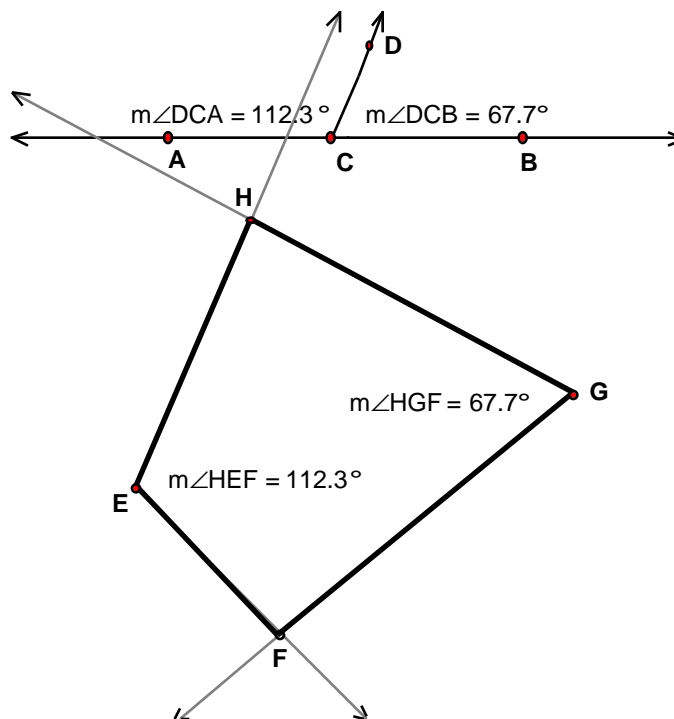
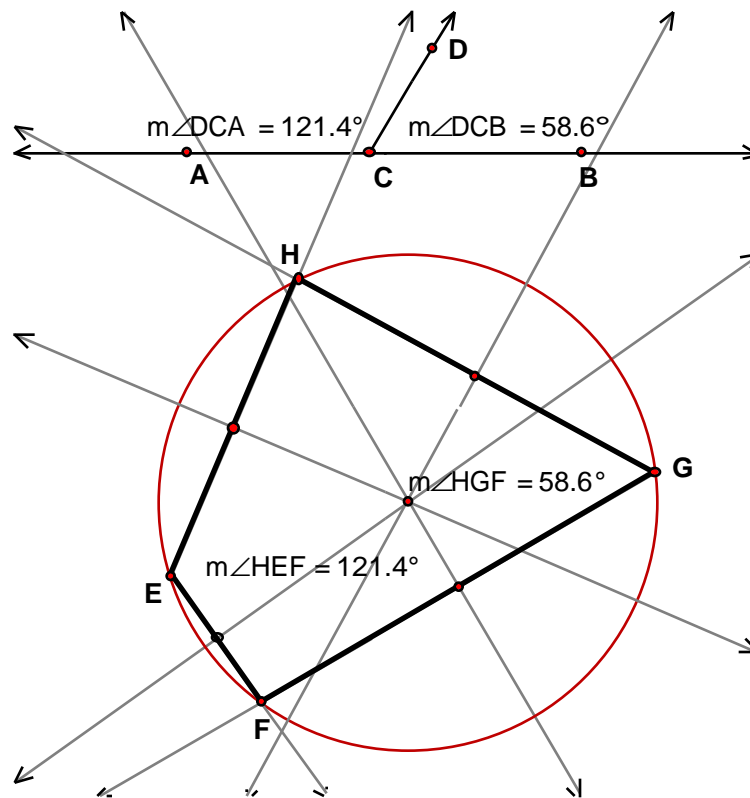


Figure 3

By repeating the same procedure, an angle  $HGF$  exactly equal to angle  $DCB$  can be constructed as shown in Figure 3, and the intersection of the rays from  $E$  and  $G$  therefore gives us the required dynamic quadrilateral  $EFGH$  with opposite angles supplementary. The size of the supplementary angles  $HEF$  and  $HGF$  can now easily be controlled by dragging point  $D$ . But how can we now check whether this quadrilateral  $EFGH$  is always cyclic irrespective of how these angles are changed?

Recall that in order for a circle to be drawn through any number of points, there has to be a central point, which is *equidistant* from all these points (i.e. the radii from this centre to all the points need to be constant). Since a perpendicular bisector is the locus of all points equidistant from two points, this obviously implies that for any polygon to be cyclic, the perpendicular bisectors of all its sides have to be concurrent in a single point for that point to be equidistant from all the vertices. Therefore to check whether  $EFGH$  is always cyclic is equivalent to checking whether the perpendicular bisectors of its sides are always concurrent as shown in Figure 4.



**Figure 4**

By dragging  $D$ , the size of the opposite supplementary angles can now be varied arbitrarily, and can it easily be checked experimentally that the perpendicular bisectors are always concurrent no matter how  $D$  is dragged. Since this point of

concurrency is equidistant from all four vertices, the unique circle passing through all four vertices can also easily be drawn by centring it at this point of concurrency.

### Mathematical Proof

So clearly the converse is true for a (convex) quadrilateral and the conjecture now achieves the status of a theorem. Of course, no amount of empirical evidence, no matter how convincing it may be, constitutes a mathematical proof of a theorem, which requires that it be demonstrated by logic alone. So how do we prove it mathematically?

#### *Theorem*

A quadrilateral  $ABCD$  is cyclic if it has a pair of opposite (or alternate) angles supplementary.

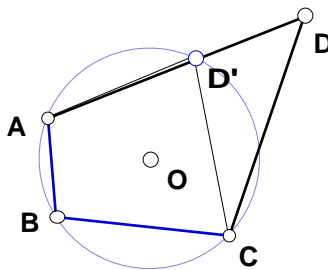


Figure 5

#### *Proof*

This sort of result is ideally suited for introducing students to a very useful form of proof called “*Proof by Contradiction*”. In this kind of proof, we start by assuming our conclusion is false. Then we show this leads to a contradiction, indicating that our conclusion must have been true. So we start by assuming the opposite angles of convex quadrilateral  $ABCD$  are supplementary, but quadrilateral  $ABCD$  is *not* cyclic.

Since any circle can be drawn through three non-collinear points, we start by drawing the circumcircle of  $ABC$ , assuming the circle does not pass through  $D$ , but through  $D'$  on ray  $AD$ . Since  $ABCD'$  is cyclic,  $\angle ABC + \angle AD'C = 180^\circ$ . But  $\angle ABC + \angle ADC = 180^\circ$  is given. Therefore,  $\angle AD'C = \angle ADC$ . From the exterior angle of a triangle theorem, we have  $\angle AD'C = \angle D'CD + \angle ADC$ . But since  $\angle AD'C = \angle ADC$ , it implies that  $\angle D'CD$  must be equal to zero. Therefore  $D$  and  $D'$  coincide, and this contradicts our initial assumption. Thus  $ABCD$  must be cyclic. (Note: If it is assumed that  $D'$  falls on  $AD$  extended, the argument is similar to the preceding, except that now  $\angle ADC = \angle DCD' + \angle AD'C$ ).

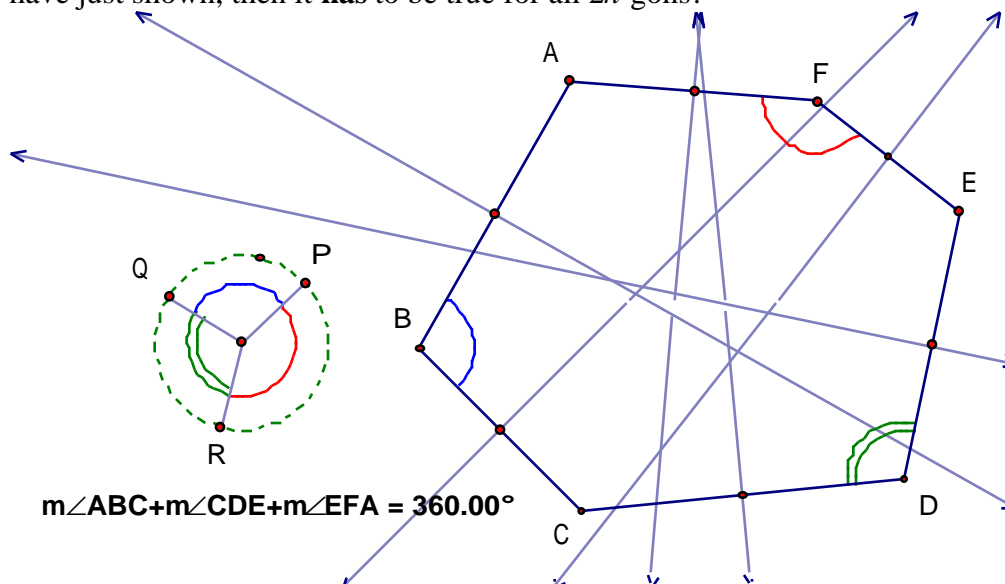
### Problems with Contradiction

Since many students at school and even at university have difficulty with proof by contradiction, it is usually helpful to point out to them that most mathematicians only use the method of "*Proof by Contradiction*" as a last resort. Generally, a direct proof is preferred (which shows that something follows directly, through the application of logical rules, from one's assumptions). One of the problems with proofs by contradiction is that they tend not to explain *why* something is true; only that it is true (because its contradiction is false). In fact, there is a small group of mathematicians called the "*intuitionists*" who reject all proofs of this kind. Their attempts, however, to rewrite mathematics without using this technique have unfortunately been largely unsuccessful. It seems therefore that there are many important results in mathematics, which unfortunately can only be proved by means of proof by contradiction.

### Further Exploration

Now what about cyclic hexagons, cyclic octagons, and cyclic  $2n$ -gons in general? Is it generally true that if a sum of alternate angles of a  $2n$ -gon is equal to  $(n - 1)180^\circ$ , then it is cyclic?

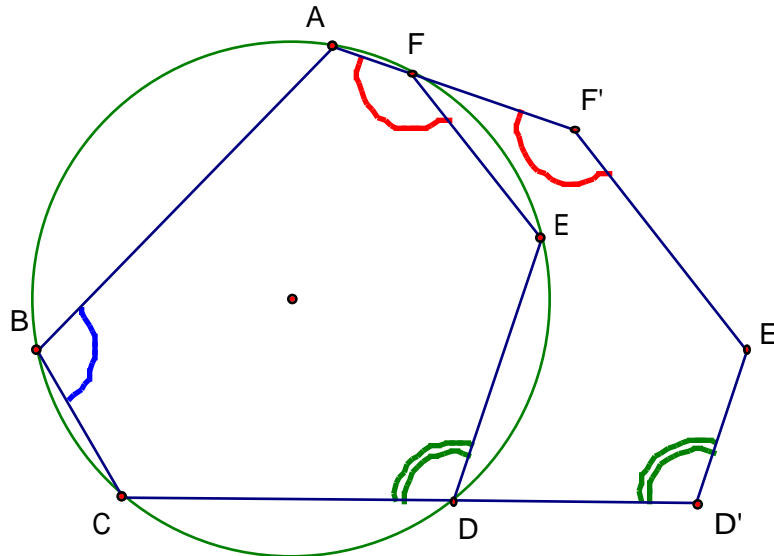
During my teaching experience with mostly prospective or in-service high school teachers, it is seldom that any doubts are exhibited about it being generally true. Typical comments are something like: "If it is true for quadrilaterals, which we have just shown, then it **has** to be true for all  $2n$ -gons!"



**Figure 6**

Indeed there is often a tangible sense of annoyance, exasperation or boredom in the class when asked to check whether it is indeed true for, say, a hexagon. Nevertheless, encouraging them to check eventually leads to some of them making the startling

discovery that it is not necessarily true for a hexagon. Starting with three angles adding up to  $360^\circ$ , it is not difficult to construct a dynamic hexagon with three alternate angles correspondingly equal to these angles. For example, consider the hexagon  $ABCDEF$  shown in Figure 6 where  $\angle B + \angle D + \angle F = 360^\circ$ , but the hexagon is not cyclic since the perpendicular bisectors of its sides are not concurrent! So there is no equidistant point in relation to all six vertices, and therefore no circle can be drawn through all six vertices!



**Figure 7**

Of interest here is also an argument given by Werner Olivier, the top student in my 2005 geometry class for prospective high school teachers. Instead of a dynamic hexagon as described above, consider any cyclic hexagon as shown in Figure 7. Respectively extend  $AF$  and  $CD$  to  $F'$  and  $D'$ , and then draw lines through these points respectively parallel to  $FE$  and  $DE$  to intersect in  $E'$ . Clearly, all the angles of  $ABCD'E'F'$  are the same as that of  $ABCDEF$ , but it is obviously not cyclic, and therefore a counter-example. Now that's the sign of mature mathematical reasoning! No wonder this student got 97% in the final examination!

One of the problems with the traditional Euclidean approach to geometry is that there are very few cases where the converses of theorems are false, and students inevitably assume that they are also always true. It is therefore a valuable strategy to go beyond dealing only with the special cases of triangles and quadrilaterals as in Euclid, but to regularly examine analogous cases for polygons where appropriate. This often provides ample opportunities for showing the difference between a

statement and its converse, and often highlights the "specialness" of triangles and quadrilaterals. Moreover, genuine mathematical research involves both proving and disproving, and both these need to be reflected in our teaching. It is simply not sufficient to just focus on developing students' skills in proving true statements, but not provide instructive opportunities for also developing their ability to find counter-examples.

### Crossed Cyclic Polygons

In mathematics, solutions to problems and answers to questions almost always lead to new problems and questions. The following questions could be used to challenge more able high school or undergraduate students. What happens if we have a *crossed cyclic* quadrilateral as shown in the first figure in Figure 8? Here the two sums of alternate (opposite) angles are clearly no longer equal, nor are they necessarily constant and can change as some of the vertices are dragged. However, if we use (directed) arc angles as shown in the second figure in Figure 8, then the two sums of alternate angles are not only equal, but remain constant no matter which vertices are dragged! (Note that this requires devising a more precise definition of what is meant exactly by the *internal angles* of a crossed polygon).

$$m\angle DAB + m\angle BCD = 111.4^\circ$$

$$m\angle ABC + m\angle CDA = 66.0^\circ$$

$$(m \text{ arc } BAD) + (m \text{ arc } BCD) = 360.0^\circ$$

$$(m \text{ arc } CBA) + (m \text{ arc } ADC) = 360.0^\circ$$

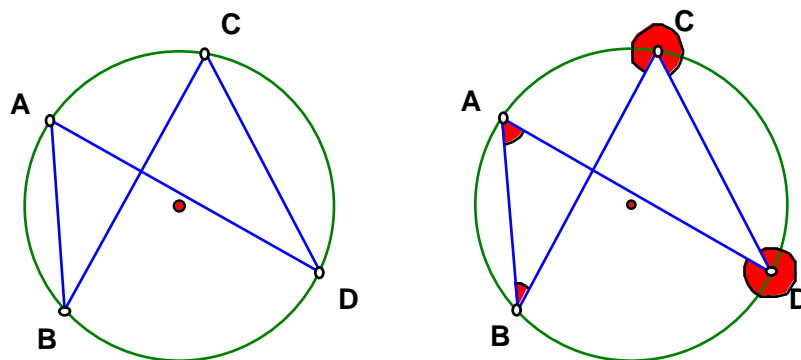
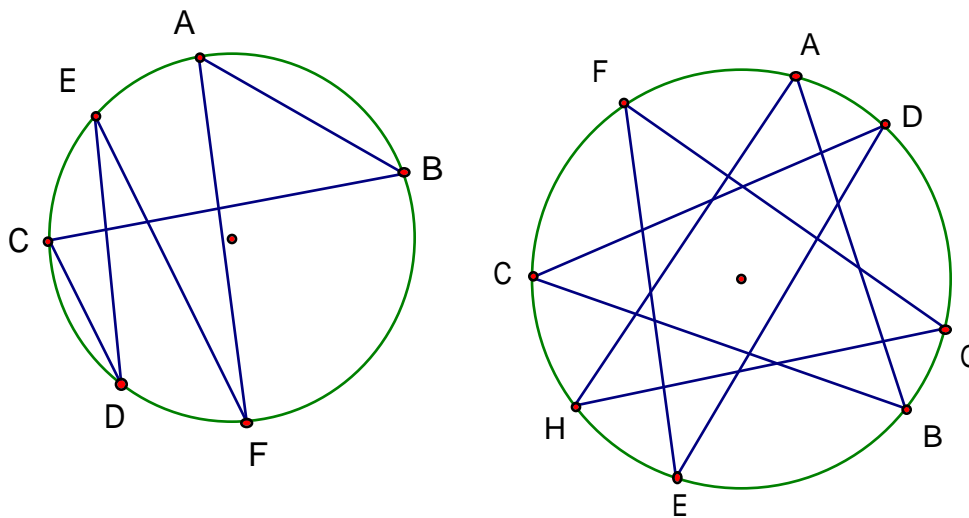


Figure 8

Are the two sums of alternate angles of any crossed cyclic polygon always equal (and constant)? The answer is "no", as shown by the crossed hexagon in the first figure in Figure 9 (even if directed arc angles are used). However, it is true for certain classes of crossed cyclic polygons such as the one shown in the second figure in Figure 9.

$$\begin{array}{llll}
 m\angle ABC = 40.21^\circ & m\angle FAB = 53.24^\circ & m\angle ABC = 52.06^\circ & m\angle HAB = 52.09^\circ \\
 m\angle CDE = 21.43^\circ & m\angle BCD = 74.09^\circ & m\angle CDE = 36.14^\circ & m\angle BCD = 42.78^\circ \\
 m\angle EFA = 18.78^\circ & m\angle DEF = 20.85^\circ & m\angle EFG = 47.34^\circ & m\angle DEF = 38.35^\circ \\
 m\angle ABC + m\angle CDE + m\angle EFA = 80.41^\circ & & m\angle GHA = 44.46^\circ & m\angle FGH = 46.77^\circ \\
 & & m\angle ABC + m\angle CDE + m\angle EFG + m\angle GHA = 180.00^\circ & \\
 m\angle FAB + m\angle BCD + m\angle DEF = 148.18^\circ & & m\angle HAB + m\angle BCD + m\angle DEF + m\angle FGH = 180.00^\circ & 
 \end{array}$$



**Figure 9**

A more complete discussion with proofs and more generalisations of the examples in Figures 8 and 9 are given in De Villiers (1996, 2003). Of interest too may be the discussion in these two books of a dual to the cyclic polygon results referred to here, which involve the sums of alternate *sides* of *circum* polygons (polygons circumscribed around a circle).

**Note**

*Sketchpad* sketches for exploring and demonstrating the results explored here can be downloaded in zipped format from:

<http://mysite.mweb.co.za/residents/profmd/cyclicpoly.zip>

These dynamic geometry figures can also be viewed and manipulated with the Free Demo of *Sketchpad* which can be downloaded from:

<http://www.keypress.com/sketchpad/sketchdemo.html>

**References**

De Villiers, M. (1996). *Some Adventures in Euclidean Geometry*. University of Durban-Westville: South Africa.



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De Villiers, M. (2003). *Rethinking Proof with Sketchpad*, Key Curriculum Press, Emeryville: USA. (A sample of zipped *Sketchpad* sketches from a few activities from the book can be directly downloaded from:

<http://mysite.mweb.co.za/residents/profmd/rethink2.zip> )

Pritchard, C. (2005). A Cyclical Property of Cyclic Polygons. *Mathematics in School*, November, 34(5), p. 14.

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**Author:** Michael de Villiers, Mathematics Education, University of KwaZulu-Natal, Ashwood 3605, South Africa. E-mail: [profmd@mweb.co.za](mailto:profmd@mweb.co.za);  
Homepage: <http://mysite.mweb.co.za/residents/profmd/homepage4.html>