Constructions for Sketchpad

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These constructions are recipes — no proofs are given. Some of them can no doubt be improved upon. Many are based on solutions in H. Dörrie, *100 Great Problems of Elementary Mathematics,* (trans. D Antin), Dover Publications (1965). Some of those on section 6 below are from T. Lemoyne, *Recherches de Géométrie Comtemporaine, Tome II, Géométrie Analytique Sans Équations*, Blanchard (1968). A few are of my own devising.

Warnings:
(a) If a construction uses the intersections of a line and a circle Sketchpad does not always distinguish correctly between them. For safety, given the line and circle, put a point P at one intersection (without using the Construct/Intersection commands). Construct a perpendicular from the centre of the circle to the line, and reflect P in the perpendicular.
(b) If a point P depends upon the position of a point X that moves on a line l, the Construct/Locus command will not always give a complete locus: Sketchpad keeps X within the borders of its notional page. It is better to start with an arbitrary circle with centre Y, say, then put X′ on the circumference, and let YX′ cut l in X. The locus is generated as X′ moves round the circle, and will be complete.

Note: Sketchpad allows points to be placed on conics and other loci, and to move along them. However, it will not construct the intersections of lines (or circles or other conics) with conics: they have to be found by Euclidean methods. I have not yet found a construction for all four intersections of two conics (or a circle and a conic). If two or three of the intersections are given, the rest can be constructed.

In general, capital letters are points, small letters are lines. AB is the line through A and B; ab is the point of intersection of a and b. AB.CD is the intersection of lines AB and CD; ab.cd is the line joining the points ab and cd.

1. The harmonic construction
Given points A, B, P on a line, to find Q such that P, Q divide A, B harmonically; i.e. AP/PB = – AQ/QB.
Take point C not on the line, and D on PC. Let E = AD.BC, F = BD.CA, Q = EF.AB, then Q is the required point.

2. The harmonic polar of a point P with respect to triangle ABC.
Let D = AP.BC, E = BP.CA, F = CP.AB; then L = EF.BC, M = FD.CA, N = DE.AB. Points L, M, N are collinear, lying on the harmonic polar of P. Obviously we need find only two of L, M, N to fix the line.

3. The harmonic pole of a line l wrt triangle ABC (sides a, b, c; a opposite A, etc.)
Let P = l.BC, Q = l.CA, R = l.AB, then the lines from A to BQ.CR, B to CR.AP, C to AP.BQ meet in the required point.
(We could say: let d = al.bc, e = bl.ca, f = cl.ab, then p = ef.bc, q = fd.ca, r = de.ab, and the pole of the common point of p, q, r. Compare this with #2 above.)

4. To find the double points of an involution

(a) Involution on a circle.

Let A, A′, B, B′ lie on a circle, the pairs (A, A′), (B, B′) defining an involution. (I.e. if AA′.BB′ is Z, then every line through Z cuts the circle in a pair of points related by this involution. A double point is a point that corresponds to itself, so in general there are two double points — the points of contact of tangents from Z to the circle).

This definition gives a construction, but the following is often more 'compact'.

Let X = AB′.A′B, Y = AB.A′B′. Then XY cuts the circle in the required points.

If XY does not cut the circle, the double points are imaginary.

If we are given C and require its mate C′ in the involution, let W = AC.XY, then A′W cuts the circle in C′.

(b) Involution on a line.

Given A, A′, B, B′ on a line. Draw an arbitrary circle (for instance, passing through A, A′) and take a point P on the circle but not on the line. Let PA, PA′, PB, PB′ cut the circle in A_1, A_1', B_1, B_1'. Use construction (a) to find the double points D_1, D_2 on the circle, and project them back, with centre P, on to the line.

5. Construction of conics

A conic can be constructed if five elements — points on the curve or lines tangent to it — are given.

5(1) Given five points A, B, C, D, E

Let l be a variable line through E. (Draw a circle centre E with any radius. Let L be an arbitrary point on the circle, and take l to be the line EL.)

The line joining AB.DE and BC.l cuts CD in Z, then P = AZ.l lies on the conic.

As L moves round the circle, P traces the conic. (Select L and P, and Construct/Locus.)

5(2) Given four points A, B, C, D and a tangent e.

(a) If e passes through D:

Let an arbitrary line l pass through A. Draw line p joining AB.e and CD.l ; then the line through D and p.BC cuts l in P on the conic.

Again, as l rotates round A, P traces the conic.

(If e passes through some other named point, relabel the points as above.)

(b) If e does not pass through a named point:

Let AB, CD, BC, DA cut e in F, F′, G, G′.

Find a double point E of the involution (F, F′), (G, G′) — const. 4(b) —

then the conic passes through A, B, C, D, E — const 5(1).

Since there are two double points, there are two possible conics, though both may be imaginary.

5(3) Given three points A, B, C and two tangents d, e.

(a) If d passes through A and e passes through B:

Draw an arbitrary line l through A. Line p joins d.BC with el; then the line through p.AB and C cuts l at P on the conic.

(b) If d passes through A but e does not pass through any of the given points:

Let X = BC.d, Y = BC.e and find a double point Z of the involution (B, C), (X, Y).

The line AZ cuts e at D on the conic. We now have points A, B, C, D and the tangent e through D, so we can use 5(2)(a). There are two double points, thus two solutions.

(c) If neither tangent passes through a given point:
Let X = BC.d, Y = BC.e and find a double point Z of the involution (B, C), (X, Y).
Let X′ = CA.d, Y′ = CA.e and find a double point Z′ of (C, A), (X′, Y′).
Now D = d.ZZ′, E = e.ZZ′ lie on the conic, and we have five points, 5(1).
Since there are two possible points Z and two possible Z′, there are four possible solutions, although some may be imaginary.

The other cases are the duals of these: we could replace point A with line a, etc. This would give the conic as the curve swept out by a moving line p. Sketchpad will do this, but the lines tend to obscure the rest of the diagram. So, instead, we find the contact points of the tangents and the curve.

5(4) Given five lines a, b, c, d, e.
Let A′ = cd, B′ = de, C′ = ea, D′ = ab, E′ = bc
The line from A′ through B′D′.C′E′ cuts a in A on the conic, and so on , cyclically.
We now have five points A, B, C, D, E on the curve — 5(1)

5(5) Given four tangents a, b, c, d and a point E.
(a) E lies on line d:
Take an arbitrary point X on a. Let cd.X cut ab.E in B, and bc.B cut d in Y, then XY is tangent to the conic.
The line through bc and the meet of ab.Y and cd.X meets XY in a point P on the curve. (NOTE: To be sure of generating the complete curve, let X′ be a movable point on a circle centre cd, and the line cd.X′ cut a in X. Then as X′ moves round the circle, XY envelopes the conic, and P traces it. Select X′ and P, and Construct/Locus to draw the conic.)
Note too that the intersection of ab.Y and cd.X traces a conic touching b at ab, and c at cd, and passing through E, as in 5(2)(a).

(b) E is not on a given tangent:
Draw a circle whose circumference passes through E. Let E.ab, E.cd, E.bc, E.da cut the circle in P, P′, Q, Q′, and find the double points D_1, D_2 of the involution (P, P′), (Q, Q′). Then ED_1 is a tangent to one possible conic, and ED_2 is tangent to a second. In either case we have 5 tangents and can use 5(4).

5(6) Given three tangents a, b, c and two points D, E.
(a) If D lies on a, E on b.
Take an arbitrary point X on a. Let bc.D cut XE in Q, and ab.Q cut c in P, then XR is a tangent.
Take an arbitrary X′ on b. Let ac.E cut X′D in Q′, and ab.Q′ cut c in R′, then X′R′ is a tangent.
We now have five tangents and can use 5(4).

(b) If D lies on a but E is not on a given tangent:
Draw a circle whose circumference passes through bc.
Let bc.D and bc.E cut the circle in P, P′, and b, c cut it in Q, Q′.
Let S, S′ be the double points of the involution (P, P′), (Q, Q′) — i.e. let R = PQ.P′Q′, R′ = PQ′, P′Q, then RR′ cuts the circle in S, S′.

Then bc.S cuts a in a point on the tangent through E. we now have four tangents and a point on one of them, and 5(5)(a) applies.
The point S′ gives a second solution.

(c) Neither point is on a given tangent:
Start as in (b) above; then bc.S passes through the intersection of tangents through D and E.
Repeat with a circle passing through ca, ca.D and ca.E giving points T, T′, and the lines c, a giving U, U′. Find the double points V, V′ of (T, T′), (U, U′), then ca.V also passes through the intersection of the tangents through D and E. We now have five tangents. The intersections of bc.S and ca.V, of bc.S′ and ca.V, of bc.S and ca.V′ and of bc.S′ and ca.V′ give four possible pairs of tangents, so four conics.

5(7) Parabolas and hyperbolas
Parabolas touch the line at infinity, which is therefore a tangent, and the above constructions can be adapted.
Hyperbolas cut the line at infinity in two points, and the tangent there are the asymptotes. If we are given the directions of the asymptotes, we effectively know two points (at infinity) on the curve. Lines, which meet in these points, are, of course, parallel.
If we are given the asymptotes themselves, we have two tangents and their points of contact (at infinity), and need one other point (or line) to determine the curve.

5(8) To draw a parabola given three points A, B, C and the direction of the axis.
Let AX, BY, CZ || axis. With a variable Q on BC, find R on BY such that QR || CA, then find P on AR such that QP || BY. As Q varies, P traces the parabola.
If D = AX.BC, and DS || CA cuts BY at S, then AS is the tangent at A.
If E = CZ.AB, and ET || CA cuts BY at T, then CT is the tangent at C.
The reflection of AX in AS and of CZ in CT intersect in the focus F.
The reflections of F in AS and CT give two points on the directrix.

5(9) To draw a conic given its foci F1, F2 and the length 2a of the major axis.
Let R be a variable point on a circle centre F_1, radius 2a, then the ⊥ bisector of F_2R cuts F_1R in P, a point on the conic, and PR is the tangent at P.

6 Lines and points associated with conics

6(1) Tangents to conic given five points A, B, C, D, E.
Let d = AB, e = BC, a = CD, b = DE, c = EA, then bd.ce cuts a in a point lying on the tangent at A.
(The points A, B, C, D, E can be in any order — sometimes a zig-zag order works better in a particular diagram.)

6(2) The points of contact of tangents to a conic, given five tangents a, b, c, d, e.
Let D = ab, E = bc, A = cd, B = de, C = ea, then the line from A through BD.CE cuts a at the point of contact of the conic.

6(3) The centre of a conic defined by points A, B, C, D, E.
Let the tangents at A, B meet at T, and those at B, C meet at T′. Let M, M′ be the midpoints of AB and BC, then the centre O is MT.M′T′.

6(4) The axes of a conic given centre O, points A, B and tangents AT, BT.
Draw the circle AOB (with centre O′) and let the line through O ∥ AT cut the circle again at A′. Let the line through O ∥ BT cut the circle again in B′. Let M = AA′.BB′, and MO′ cut the circle in U, V. The axes are OU, OV.
NOTE: in practice, when finding A′ it is safer to construct the line through O ∥ AT, drop ⊥ from O′ to this line, and reflect O in this ⊥. Find B′ similarly.
For U, V, label one intersection U, then V is O′ translated by the vector UO′ — select O′ then U, Transform/Mark Vector; then select O′ again and Transform/Translate/By marked vector.)

6(5) To find the asymptotes of a hyperbola
In this case M, in 6(4) lies outside the circle AOB. The tangents from M to the circle are the asymptotes. So draw a segment MO′ and find its midpoint. With this centre draw a circle through M to cut the original circle at T, T′ (find T, reflect in MO′ for T′). OT, OT′ are the asymptotes.

6(6) To find the foci
Let the tangent and normal at A cut the major axis at A′ and A″, and draw the circle with diameter A′A″, letting its centre, the midpoint of A′A″, be A*. Construct a tangent from O to this circle (the circle with OA* as diameter cuts it at the point of tangency, K).
The circle with centre O and radius OK cuts the major axis at one focus F_1. The second focus is best found either as O translated by vector F_1O, or as the reflection of F_1 in the minor axis.

6(7) To find the ends of the axes
Let F* be the foot of the ⊥ from F_1 to the tangent at A. The circle, centre O radius OF* is the auxiliary circle, and cuts the major axis at the vertices of the conic.
The circle, centre F_1 and radius OF* cuts the minor axis where it intersects the conic.

6(8) To find the directrices
(a) Ellipse: Let the line through F_1 ⊥ major axis cut the auxiliary circle at L; then the line through L ⊥ OL cuts the major axis at a point on the directrix. The directrix is ⊥ major axis. The second directrix is the reflection of the first in the minor axis.

(b) Hyperbola: The tangents from F_1 to the auxiliary circle touch the circle at points on the directrix. (The circle with diameter OF_1 cuts the auxiliary circle in the required points.) Reflect this directrix in the minor axis to find the second directrix.

6(9) The radius and centre of curvature at a point A on the conic.
Construct the normal at A (⊥ tangent) and let it cut the major axis at A″
Line through A″ parallel to the tangent cuts AF_1 at Q.
Line through Q ∥ AF_1 cuts the normal at R, the centre of curvature.
AR is the radius of curvature.

6(10) Given two conics through A, B, C, to find the fourth intersection.
Let the tangents to one conic at A, B meet at T, and the tangents to the other meet at T′. Let D = TT′.BC, E = TT′.CA, then the fourth intersection is AD.BE.

6(11) Given line l cutting a conic at A, to find the second intersection.
Take four arbitrary points B, C, D, E on the conic, in any order such as A, E, C, B, D. Let X = AB.DE, Y = CD.l, Z = XY.BC, then the intersection is EZ.l.

6(12) To find double elements of projectivities defined by (A, B, C) → (A′, B′, C′).
(a) Given two triples of points (A, B, C), (A′, B′, C′) on a circle:
let (a = A′A), b = A′B, c = A′C, (a′ = AA′), b′ = AB′, c′ = AC′; then the line bb′.cc′ cuts the circle in the required double points H, K.
Note: HK is the Pascal line of the hexagon AB′CA′BC′.

(b) Given two projective pencils of lines with a common centre P:
draw a circle through P and let a, b, c, … cut the circle in A, B, C, … . Find the points H, K as in (a), then the lines PH, PK are the double elements.

(c) Given two triples of points on a line:
Take an arbitrary point P not on the line, and draw a circle through it. The rays from P to the points cut the circle as in (a). That construction gives H, K; and PH, PK cut the line in the required points.

6(13) The points of intersection of a line l with a conic through A, B, C, D, E.
Let DA, DB, DC cut l in α, β, γ; and EA. EB EC cut it in α′, β′, γ′; and proceed as in 6(12) above. I.e. take a point not on l, such as D, and draw a circle passing through D. Let DA, DB, DC cut the circle in A_1, B_1, C_1, and Dα′, Dβ′, Dγ′ cut it in A_2, B_2, C_2. The line joining $A_1B_2.A_2B_1$ and $A_1C_2.A_2C_1$ cuts the circle in H and K, and DH, DK cut l in the required points.

6(14) The tangents from P to a conic determined by tangents a, b, c, d, e.
(If the conic is determined by 5 points we must construct the tangent at each point, or use 6(15) below.)
Take points α = ad, β = bd, γ = cd, and α′ = ae, β′ = be, γ′ = ce. Draw a circle passing through P and let Pα, Pβ, Pγ cut it in A_1, B_1, C_1; Pα′, Pβ′, Pγ′ cut it in A_2, B_2, C_2.
If the line joining $A_1B_2.A_2B_1$ and $A_1C_2.A_2C_1$ cuts the circle in H and K, then PH and {K are the required tangents. The points of contact can be determined from, say, the five tangents PH, PK, a, b, c.

6(15) The polar of a point P with respect to a conic.
Let A, B be points on the conic, and PA, PB cut the conic again in A′, B′. The polar is the line joining AB.A′B′ and AB′.A′B. (This line cuts the conic in points on the tangents from P to the conic.)

6(16) To find the pole of a line l with respect to a conic.
Preliminary: If l is tangent to a conic K and passes through P, to find the second tangent through P.
Let a, b, c, d be four arbitrary tangents to K and let Q be the intersection of al.cd and P.bc. Then ab.Q cuts d in a point on the second tangent.
Main: Let a, b be two tangents to the conic, and find a′, the second tangent through al; and b′, the second through bl. The pole is the intersection of ab.a′b′ and ab′.a′b.
(or we can join the contact points of a and a′, and of b and b′. These intersect in the pole.)

7 Miscellaneous constructions related to conics

7(1) A parabola through given points A, B, C, D.

Take an arbitrary point Z and draw lines p, q, p′, q′ through it || AB, BC, CD, DA.
Construct the double rays of the involution (p, p′), (q, q′).
[Let the rays cut a circle through Z in P, P′, Q, Q′, then the line joining PQ. P′Q′ and PQ′.P′Q cuts the circle in X, Y, say. There are two parabolas, axes || ZX and || ZY.]
Now construct the parabola through 3 points given the direction of the axis.

7(2) A rectangular hyperbola through given points A, B, C, D.

Find the orthocentre H of ΔABC. The curve passes through A, B, C, D, H.

7(3) Inscribed (tritangent) conics given triangle ABC (with sides a, b, c)

(a) Given the centre O: Reflect two sides, say b, c, in O, giving b′, c′. Construct the conic touching a, b, c, b′, c′.

(b) Given an axis l: Reflect b, c in l giving b′, c′. The conic touched a, b, c, b′, c′.

(c) Given a focus F_1: (The foci of an inscribed conic are isogonal conjugates.)
Reflect AF_1 in the bisector of angle A, and BF_1 in the bisector of angle B. The reflections meet in F_2, the second focus. The axis is F_1F_2, the centre is at its midpoint, and we can use (a) or (b).
Or if we reflect F_1 in the sides a, b, c, giving F_a, F_b, F_c, then F_2F_a cuts BC in the point of tangency. $F_2F_a = F_2F_b = F_2F_c$ = length of major axis.
Draw circle centre F_2, radius F_2F_a, and let Q be any point on it. The ⊥ bisector of F_1Q cuts F_2Q at a point P on the conic. As Q traces the circle, P traces the conic.

7(4) Conics circumscribing triangle ABC

(a) Given the centre O: Reflect two vertices, say B and C, in O giving B′, C′. The conic passes through A, B, C, B′, C′. (To construct tangent, note that BC′ || B′C, so a line through A and their intersection is || BC′.

(b) Given an axis l: Reflect B, C in l giving B′, C′, and draw conic through A, B, C, B′, C′.

(c) Given a focus F_1:
Preliminary: Given ΔABC and lines l_1, l_2, l_3, we find points T on l_1, U on l_2, V on l_3 such that UV passes through A, etc.
Let G be l_1.CA, H be l_2.BC, X be l_2.AB, Y be l_3.CA, Z be GH.XY, then V = l_3.BZ, T = l_1.BZ, U = VA.TC.
Main: Let l_1, l_2, l_3 bisect BF_1C, CF_1A, AF_1B respectively, and use the preliminary construction for T, U, V. The line from T through the midpoint of BC, and from U through the midpoint of CA meet at the centre O, and we use construction (a).
There are four conics: one uses the three internal angle bisectors as l_1, l_2, l_3, the others use one internal and two external angle bisectors.