Amesa KZN Math Journal, December 2004, Vol. 8, No. 2, pp. 34-37

Crookes & Monteith's Proof of a Classic Problem

Michael de Villiers, profmd@mweb.co.za
Mathematics Education, UKZN

In my e-Newsletter[1] in February 2004, I posted an old classic problem that seemed to have first appeared in the *Mathematical Gazette* in 1923, but had also been used in the 1916 entrance exam for Peterhouse and Sidney Sussex Colleges, Cambridge University (see Bolt, 2003). Since then it resurfaces every now and again in journals around the world. What makes the problem so interesting is that it at first glance it seems really trivial and elementary, but turns out to be deceptively more difficult than anticipated.

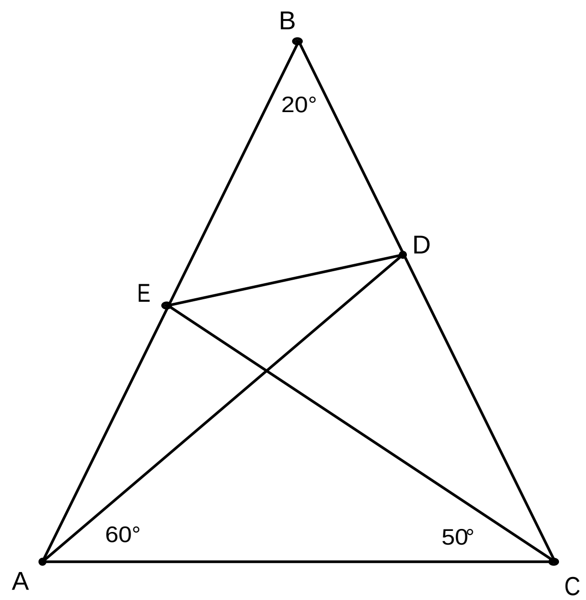

Figure 1

The problem

In an isosceles triangle *ABC* with base *AC* the angle at *B* equals 20 degrees (Figure 1). Points *D* and *E* are chosen on the sides *BC* and *AB* respectively so that angle *DAC* = 60˚ and angle *ECA* = 50˚. Find angle *ADE.*

Probably the most straightforward and quickest approach is to use some trigonometry (or co-ordinate geometry). The trick in the problem is to realise that since only angles

[1] If you'd like to receive the e-Newsletter, please send me an e-mail to my address as given above.

are given, one need only consider any specific triangle with a conveniently chosen side (since similarity is maintained and one can enlarge or reduce the size as required, but that does not affect the size of angle *ADE*).

So one could start, for example, by choosing *AB* = 10 units (= *CB*), and then calculate *AC* using the cosine rule, use the sine rule to calculate *EC*, etc. until you have enough to determine angle *ADE* (which incidentally happens to be 30 degrees). However, it is rather an inelegant and tedious approach. One could also use co-ordinate geometry by placing the triangle with vertices on the *x* - and *y*-axes. Despite the inelegance, there is some elegance in the "*trick*" which is something Mathematics Olympiad enthusiasts ought to know and use when appropriate.

Crookes & Monteith's Attempts

A short while after sending out the e-Newsletter, I got an e-mail message from two enthusiastic Grade 9 students of Sue Southwood from Hilton College, namely, Stephen Crookes and Iain Monteith. They had drawn an accurate *Sketchpad* sketch and had measured the angle *ADE* to find the correct value of 30˚. Having pointed out to them that they needed to provide a proof (or a series of calculations), they subsequently came back with the attempt shown in Figure 2 which consisted of drawing a circle with center *A* and radius *AD*, and the other segments as shown.

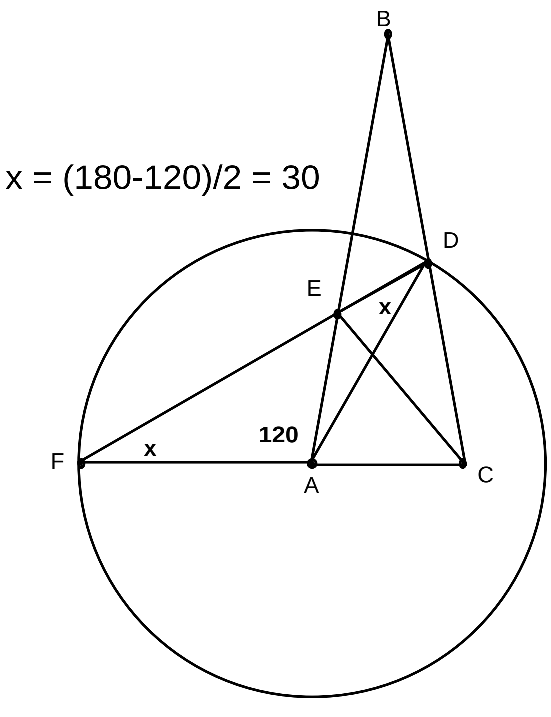

Figure 2

I then replied to them that mathematically, they still had to prove that when they drew the circle with *A* as center passing through *D* and intersecting *CA* extended at *F*, that the points *F*, *E* and *D* actually lay in a straight line (collinear). (They did so in their construction because it was accurate, but they still had to *prove* that they actually did to complete their proof). Alternatively, they could draw *DE* extended to meet *CA* extended in *F* (and therefore ensure they were in a straight line!), but then just using that information they had to *prove* that indeed *AF* = *AD* (i.e. that *AFD* is isosceles).

Much to my pleasant surprise, the two boys did not give up, and eventually sent me the ingenious proof shown in Figures 3 and 4 (that are posted unedited), and is different from the synthetic proof given in Diamond & Georgiou (2001).[2]

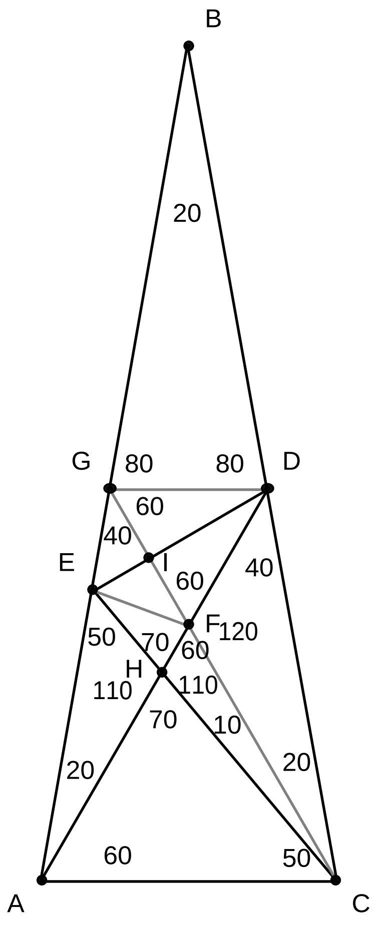

Figure 3

[2] As is often the case, the boys' proof is not original. While packing for my move to the Edgewood Campus, I came across a faded article by Constantine Knopp, Nine solutions to one problem, from the May/June 1994 issue of some unidentifiable journal which has exactly the same diagram and a similar proof.

First We constructed the Triangle with all the correct lines.
Then We constructed some lines of our own(red dashed lines).

Then we put the angles we knew in.

As you can see, Triangle ACF is equilateral
therefore AC=AF=FC

As you can see Triangle ACE is isos. because angle ACE is 50 and angle AEC is 50
Therefor AC=AE

So we can say AC=AF=FC=AE

So triangle EAF is isos. with angle AEF being 80 and angle AFE being 80

If angle AFE is 80 then angle EFG must be 40 as angle AFG is 120. So 120 - 80 = 40

Triangle GEF is isos. because angle EFG and angle EGF both = 40
Therefore EF=EG

As triangle GDF is equilateral GF=FD=DG

So Quad. EGDF is a kite

So angle EIF is 90

If angle EIF =90 then angle IDF must =30 as 180 - (90 + 60) = 30

Figure 4

References

Bolt, B. (2003). Introduction: Recreational Mathematics. In Pritchard, C. (2003). *The Changing Shape of Geometry*. Cambridge University Press.

Diamond, R.A. & Georgiou, G.R. (2001). Triangles and Quadrilaterals Revisited, Part 2: The Solution. *Mathematics in School*, 30 (1), (Nov 2002), 11-13.

****

"It's the latest innovation in office safety. When your computer crashes, an air bag is activated so you won't bang your head in frustration."