

Crookes & Monteith's Proof of a Classic Problem

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In my e-Newsletter¹ in February 2004, I posted an old classic problem that seemed to have first appeared in the *Mathematical Gazette* in 1923, but had also been used in the 1916 entrance exam for Peterhouse and Sidney Sussex Colleges, Cambridge University (see Bolt, 2003). Since then it resurfaces every now and again in journals around the world. What makes the problem so interesting is that at first glance it seems really trivial and elementary, but turns out to be deceptively more difficult than anticipated.

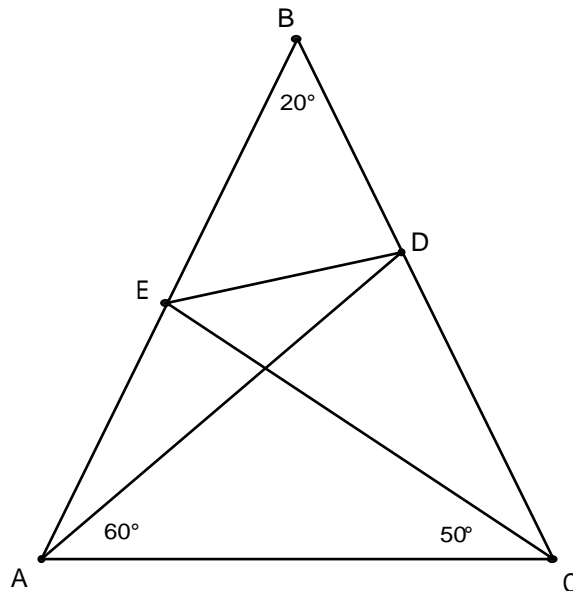


Figure 1

The problem

In an isosceles triangle ABC with base AC the angle at B equals 20 degrees (Figure 1). Points D and E are chosen on the sides BC and AB respectively so that angle $DAC = 60^\circ$ and angle $ECA = 50^\circ$. Find angle ADE .

Probably the most straightforward and quickest approach is to use some trigonometry (or co-ordinate geometry). The trick in the problem is to realise that since only angles

¹ If you'd like to receive the e-Newsletter, please send me an e-mail to my address as given above.

are given, one need only consider any specific triangle with a conveniently chosen side (since similarity is maintained and one can enlarge or reduce the size as required, but that does not affect the size of angle ADE).

So one could start, for example, by choosing $AB = 10$ units ($= CB$), and then calculate AC using the cosine rule, use the sine rule to calculate EC , etc. until you have enough to determine angle ADE (which incidentally happens to be 30 degrees). However, it is rather an inelegant and tedious approach. One could also use coordinate geometry by placing the triangle with vertices on the x - and y -axes. Despite the inelegance, there is some elegance in the "trick" which is something Mathematics Olympiad enthusiasts ought to know and use when appropriate.

Crookes & Monteith's Attempts

A short while after sending out the e-Newsletter, I got an e-mail message from two enthusiastic Grade 9 students of Sue Southwood from Hilton College, namely, Stephen Crookes and Iain Monteith. They had drawn an accurate *Sketchpad* sketch and had measured the angle ADE to find the correct value of 30° . Having pointed out to them that they needed to provide a proof (or a series of calculations), they subsequently came back with the attempt shown in Figure 2 which consisted of drawing a circle with center A and radius AD , and the other segments as shown.

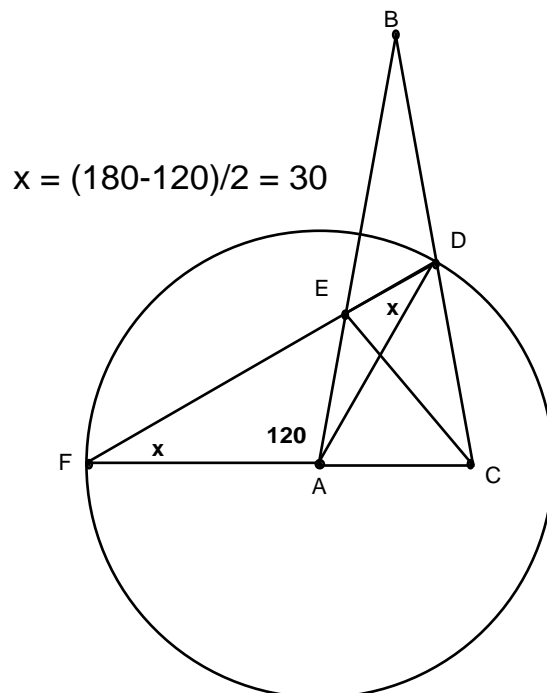


Figure 2

First We constructed the Triangle with all the correct lines.
Then We constructed some lines of our own (red dashed lines).
Then we put the angles we knew in.

As you can see, Triangle ACF is equilateral
therefore $AC=AF=FC$

As you can see Triangle ACE is isos. because angle ACE is 50 and angle AEC is 50
Therefore $AC=AE$

So we can say $AC=AF=FC=AE$

So triangle EAF is isos. with angle AEF being 80 and angle AFE being 80

If angle AFE is 80 then angle EFG must be 40 as angle AFG is 120. So $120 - 80 = 40$

Triangle GEF is isos. because angle EFG and angle EGF both = 40
Therefore $EF=EG$

As triangle GDF is equilateral $GF=FD=DG$

So Quad. EGDF is a kite

So angle EIF is 90

If angle EIF = 90 then angle IDF must = 30 as $180 - (90 + 60) = 30$

Figure 4

References

- Bolt, B. (2003). Introduction: Recreational Mathematics. In Pritchard, C. (2003). *The Changing Shape of Geometry*. Cambridge University Press.
- Diamond, R.A. & Georgiou, G.R. (2001). Triangles and Quadrilaterals Revisited, Part 2: The Solution. *Mathematics in School*, 30 (1), (Nov 2002), 11-13.

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