Theorem: The perpendicular bisectors of the sides of a circum quad (a quadrilateral circumscribed around a circle) form another circum quad.

Proof

We shall use the notation given in the figure above and also the following:
AB=a, BC=b, CD=c, DA=d, \( t_A \) is the tangent from A to the incircle, \( r \) is the radius of the incircle. We shall assume that \( r=1 \). We now have:

\[
MD = -\frac{d}{2 \cos D}, \quad MW = \frac{c}{2} - \frac{d}{2 \cos D},
\]

\[
FW = MW \cot(180°-D) = \frac{d}{2 \sin D} - \frac{c}{2} \cot D.
\]

Similarly, \( FX = \frac{c}{2 \sin D} - \frac{d}{2} \cot D \) and hence:
2(FX - FW) = \frac{c - d}{\sin D} + (c - d) \cot D

= (c - d) \frac{1 + \cos D}{\sin D}

= (c - d) \cot \frac{D}{2}

= (c - d) \frac{t_D}{r}

= (c - d)t_D.

Similarly for 2(HZ-HY), 2(EY-EX) and 2(GW-GZ). We now have to check that the sum of the opposite sides are equal, or alternatively that EF-EH+HG-GF=0, i.e. that (FX-EX)-(HY-EY)+(HZ-GZ)-(FW-GW)=0 or 2(FX-FW)+2(HZ-HY)+2(EY-EX)+2(GW-GZ)=0. In view of the previous results, the last equality is equivalent to:

(1)... (c - d)t_D + (d - a)t_A + (a - b)t_B + (b - c)t_C = 0.

Since $c - d = b - a$ and $d - a = c - b$, equation (1) is equivalent to:

(2)... (c - d)(t_D - t_B) + (d - a)(t_A - t_C) = 0.

But $c - d = (t_C + t_D) - (t_B + t_A) = t_C - t_A$ and similarly $d - a = t_D - t_B$.

Therefore (2) is equivalent to $(t_C - t_A)(t_D - t_B) + (t_D - t_B)(t_A - t_C) = 0$, which is an identity and completes the proof.

**Note:** Download a Zipped Sketchpad sketch directly from

http://mysite.mweb.co.za/residents/profmd/circumquad.zip