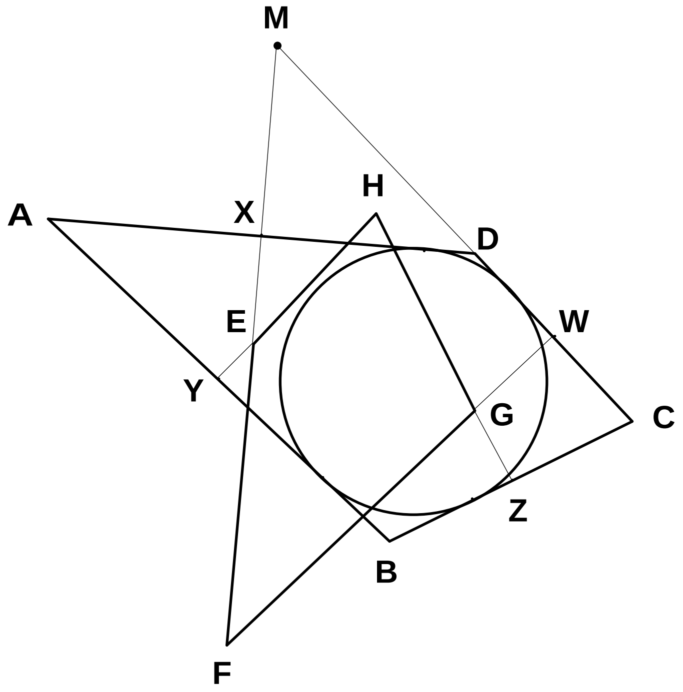
From: De Villiers, M. (1994; revised 1996). *Some Adventures in Euclidean Geometry*, Univ. of Durban-Westville (now Univ. of KwaZulu-Natal), pp. 192-193.

Theorem: The perpendicular bisectors of the sides of a circum quad (a quadrilateral circumscribed around a circle) form another circum quad.



Proof

We shall use the notation given in the figure above and also the following: AB=*a*, BC=*b*, CD=*c*, DA=*d*, t_A is the tangent from A to the incircle, *r* is the radius of the incircle. We shall assume that *r*=1. We now have:

$$MD = -\frac{d}{2\cos D},\ \ MW = \frac{c}{2} - \frac{d}{2\cos D},$$

$$FW = MW\cot(180° - D) = \frac{d}{2\sin D} - \frac{c}{2}\cot D.$$

Similarly, $FX = \dfrac{c}{2\sin D} - \dfrac{d}{2}\cot D$ and hence:

$$2(FX - FW) = \frac{c-d}{\sin D} + (c-d)\cot D$$
$$= (c-d)\frac{1+\cos D}{\sin D}$$
$$= (c-d)\cot\frac{D}{2}$$
$$= (c-d)\frac{t_D}{r}$$
$$= (c-d)t_D.$$

Similarly for 2(HZ-HY), 2(EY-EX) and 2(GW-GZ). We now have to check that the sum of the opposite sides are equal, or alternatively that EF-EH+HG-GF=0, i.e. that (FX-EX)-(HY-EY)+(HZ-GZ)-(FW-GW)=0 or 2(FX-FW)+2(HZ-HY)+2(EY-EX)+ 2(GW-GZ)=0. In view of the previous results, the last equality is equivalent to:

(1)... $(c-d)t_D + (d-a)t_A + (a-b)t_B + (b-c)t_C = 0.$

Since $c - d = b - a$ and $d - a = c - b$, equation (1) is equivalent to:

(2)... $(c-d)(t_D - t_B) + (d-a)(t_A - t_C) = 0.$

But $c - d = (t_C + t_D) - (t_D + t_A) = t_C - t_A$ and similarly $d - a = t_D - t_B$.

Therefore (2) is equivalent to $(t_C - t_A)(t_D - t_B) + (t_D - t_B)(t_A - t_C) = 0$, which is an identity and completes the proof.

Note: Download a Zipped Sketchpad sketch directly from

http://mysite.mweb.co.za/residents/profmd/circumquad.zip